Existence of Equilibrium in All-Pay Auctions with Price Externalities*

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Abstract

This paper investigates all-pay auctions with general price externalities and complete information. We show the existence of a mixed strategy Nash equilibrium by using Schauder's fixed-point theorem. The Brouwer's fixed-point cannot be applied because of the infinite-dimensional set of distribution functions. Our findings are applicable to future works on contests and charity auctions.

KEYWORDS: All-pay auction, contest, externalities

JEL CLASSIFICATION: C72, D44, D62

1 Introduction

All-pay auctions are used both as an auction device and a contest. In this game, all bidders have to pay their bids and the one who submitted the highest bid wins. Fundraising mechanisms and race competitions are two of the many applications. Competitors on a race care about the recognition they could get from their participation. Therefore, a loser is better off with an effort closer to the winner's effort and a winner with an effort further to the highest loser's effort. In a charity auction, bidders usually care about the charity purpose and benefit from the total amount raised during the auction. In both situations, participants benefit from a price externality, either dependent or independent of the winner identity.[1]

The most famous result in this literature is that all-pay auctions are optimal mechanisms at raising money for charity (Goeree et al. (2005) and Engers and McManus (2007)). However, this is not confirmed on the field. Carpenter et al. (2008) and Onderstal et al. (2013) observe

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[1] Notice there is also a large literature about auctions with allocative externalities initiated by Jehiel and Moldovanu (1996) with complete information. The key in this literature is the winner identity and not, as here, the money/efforts spent. See Klose and Kovenock (2015) for an analyzis of the all-pay auction with allocative externalities.

a low participation in field experiments and determine other better fundraising mechanisms. That might be due to the assumption of linear externality that participants benefit.[2] Indeed, all theoretical results about charity auctions rely on this linearity assumption. Non linear externalities could lead to other optimal mechanisms and therefore affect the bidders' participation at the equilibrium. In this paper, we propose to investigate the all-pay auction with complete information without specifying the shape of the externality functions. Complete information is not as usual in auction theory as in the contest literature. However, there are recent works about auctions with externalities which investigate a setting with complete information.[3]

Our analysis does not make any assumptions on the analytic form of the externality function, and consider that bidders take in account a positive externality from their own bid and either a positive or a negative externality from their rivals' bids. Thus, the externality function might also reflect a statistical link between the bids of a bidder and her competitors. These are relevant with economic applications, and more specifically with charity auctions and race competitions. We establish the existence of a Nash equilibrium with mixed strategies, defined on a closed, convex and infinite-dimensional set of continuous distribution functions. Therefore we use the Schauder's fixed-point theorem to determine this result despite the infinite-dimensional set of functions. That might be useful for future research on fundraising mechanisms.

The remainder of this paper is structured as follow. Section 2 introduces the formal setting and properties of general price externalities. In Section 3 we discuss the non-existence of a pure strategy Nash equilibrium and show the existence of a mixed strategy Nash equilibrium. We conclude in Section 4. In the following we refer to the players as *bidders*, keeping the auction terminology.

2 The Model

Suppose n risk-neutral bidders submit their bids for an indivisible object (or prize) which is allocated to the highest bidder. We denote the set of the bidders by $\mathcal{N} = \{1, ..., n\}$. Bidder i's value is given by v_i such that $v_1 \geq v_2 \geq ... \geq v_n \geq 0$. Although valuations are common knowledge among the potential bidders, the seller has no information about them. All bidders have to pay their bids. Therefore, either bidder i wins the auction with a bid x_i, and obtains a payoff $v_i - x_i$, or she loses and obtains $-x_i$.[4] Moreover, the money raised from each potential bidder potentially

[2] Among the other possible explanations, Carpenter et al. (2010) suggest the unfamiliarity of the participants to the mechanism and endogenous participation (which is not considered here) and Bos (2016) heterogeneity among bidders.

[3] Among them, Jehiel and Moldovanu (1996) investigate the consequences of allocative externalities on participation decisions in the first-price winner-pay auction, Konrad (2006) examines ownership structure through an all-pay auction with externalities dependent of the firms' identity, Ettinger (2010) investigates the first-price and second-price winner-pay auctions with price externalities, Klose and Kovenock (2015) analyze the all-pay auction with allocative externalities, Bos (2016) compares the performance of all-pay auctions and winner-pay auctions for charity purpose and Damianov and Peeters (2018) compare the lowest-price all-pay auction with other fundraising mechanisms.

[4] If more than one bidder submit the same winning bid, they win with equal probability.

impacts the utility of each other: each bidder benefits from her own participation and either benefits or suffers from her rivals' bids.

Thus, the bidder's utility function includes a price externality depending on all bids paid. That could be a function of the sum of the bids $\sum_{i=1}^{n} x_i$, the revenue raised, as in charity auctions, and thus independent of the winner's identity. That could also be a function of the difference between bids contingent on the winner's identity. Indeed in contests, participants might get a reward not only from winning but also from her relative performance compared to others. Accordingly we consider an externality function that depends on all bids, $h_i : \mathbb{R}_+^n \to \mathbb{R}_+$. It follows that bidder i's utility is given by[5]

$$U_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} v_i - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is the only winner} \\ \frac{v_i}{k} - x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{if } i \text{ is one of the } k \text{ winners} \\ -x_i + h_i(x_i, \boldsymbol{x}_{-i}) & \text{otherwise} \end{cases} \tag{1}$$

with $\boldsymbol{x}_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. The linear case can lead to $\alpha_i \sum_{i=1}^{n} x_i$, with α_i a positive number, which is the usual form of the externality function in the charity auction literature[6] and other works about auctions with identity-independent price externalities[7]. For the analysis we make the following assumptions, relevant with economic applications.

Assumption 1 ($A1$)**.** *If all bidders submit a zero bid, none benefit from the externality:* $h_i(0, ..., 0) = 0$.

Assumption 2 ($A2$)**.** *The externality function is a* $\mathcal{C}^1$ *function.*

As stated by Assumption $A1$, it is relevant that non active participation of all bidders does not induce any benefit, while $A2$ is a technical assumption.

Assumption 3 ($A3$)**.** *Bidder i benefits from an increasing externality in her own bid.*

Although bidder i gets a benefit from a rise in her own bid, such as $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \geq 0$, this is not necessarily the case from her rivals' bids. This is contingent on the shape of h_i and therefore how bidder i perceives her competitors' bids and the statistical link between all bids.

Assumption 4 ($A4$)**.** *Bidder i's payoff is decreasing with her payment x_i in the auction.*

Assumption $A4$ means bidders are always better off by paying a lower price and leads to $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leq 1$. In the literature about about all-pay auctions with price externalities, it is either implicitly or explicitly assumed that the payoff cannot increase with the bidder payment, which means that $\dfrac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. Applied to charity auctions, this assumption displays the limit of the bidders' altruism. Otherwise bidders could be indifferent between giving and keeping money for their personal use.[8] In a contest, such as a race, a competitor gets a higher

[5]Bidder i might be one of the k winners, all submitting the same highest bid, such that $k = \#\{j | j = \arg\max\{x_i, i \in \mathcal{N}\}$.

[6]See Goeree et al. (2005), Engers and McManus (2007) and Bos (2016).

[7]See for example Ettinger (2010).

[8]Goeree et al. (2005) and Bos (2016) consider the linear case $\alpha \sum_{i=1}^{n} x_i$ and assume that α is strictly inferior to 1. Engers and McManus (2007) add a *warm glow* a la Andreoni (1989), making bidders more sensitive to their own bid for cognitive/psychological reasons and thus benefiting differently from their own bids and their rivals' bids. Therefore, bidder i benefits from the externality $\alpha x_i + \beta \sum_{j=1, j \neq i}^{n} x_j$ with $1 > \alpha \geq \beta > 0$.

(lower) payoff from a better (worst) relative performance, $\alpha_i(x_i - \max_{j\neq i} x_j)$, due to reputation or recognition. Yet the marginal cost has to be higher than the marginal benefit from her own performance, otherwise her effort might go to infinity. That requires again $\alpha_i < 1$, which is here $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$. Our investigation is more general. We also consider the limit case such that bidders could be fully altruistic in a charity auction and the marginal cost of the winner in a race could be equal to her marginal benefit.

3 Existence of an Equilibrium

In this section, we show the existence of a Nash equilibrium in mixed strategies. To understand better the extent of this result, we first discuss how the (non) existence of a Nash equilibrium in pure strategies is contingent on the shape of the externality functions.

It is a well known result there is no bidding equilibrium in pure strategies in all-pay auctions without price externalities (see Hillman and Riley (1989) and Baye et al. (1996)). Unsurprisingly, that is also the case for the class of externality functions such that $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) < 1$ for all $i = 1, ..., n$.

We only provide a sketch of the proof with two-bidder to make the argument easier to follow. Let us assume that $x_i \geq x_j$, then two cases may arise. First if bidder j can overbid, her profitable deviation is $x_i + \varepsilon$ for $\varepsilon > 0$, such that $v_j - (x_i + \varepsilon) + h_j(x_i, x_i + \varepsilon) \geq -x_j + h_j(x_i, x_j)$. In this case, $x_i \geq x_j$ is not possible. Second if j cannot overbid, her profitable deviation is to offer zero since $h_j(0, x_i) > -x_j + h_j(x_j, x_i)$ given $\frac{\partial h_j}{\partial x_j}(x_j, x_i) < 1$. Therefore, i's profitable deviation is to offer $\varepsilon > 0$. As a result, this is unstable and there is no pure strategy Nash equilibrium.

There are also many situations in our general case described by assumption A4, $\frac{\partial h_i}{\partial x_i}(x_i, \boldsymbol{x}_{-i}) \leq 1$ for all $i = 1, ..., n$, in which there is no Nash equilibrium in pure strategies. To make the statement more straightforward to follow, we focus again on the two-bidder case.

Let suppose both bidders maximize $v_i - x_i + h_i(x_i, x_j)$ for $i = 1, 2, i \neq j$. Then they choose $(\tilde{x}_1, \tilde{x}_2)$ such that

$$\frac{\partial h_1}{\partial x_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial h_2}{\partial x_2}(\tilde{x}_2, \tilde{x}_1) = 1.$$

If both bidders benefit from the same externality functions, $h_1 = h_2 \equiv h$, they bid $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$ and get a payoff $\frac{v_i}{2} - \tilde{x} + h(\tilde{x}, \tilde{x})$ for $i = 1, 2$. Let us now consider $v_1 > v_2$ such that bidders do not have the same maximum bid. In this case, bidder i is always better off by overbidding $x_i = \tilde{x} + \varepsilon$ for $\varepsilon > 0$ and $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$ cannot be an equilibrium.[9] Using the Taylor's theorem at the point $\tilde{x}$ such that $h(\tilde{x}+\varepsilon, \tilde{x}) = h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) + o(\varepsilon)$ with $\lim_{\varepsilon\to 0} o(\varepsilon) = 0$, a bid $\tilde{x} + \varepsilon$ leads to the payoff $v_i - \tilde{x} - \varepsilon + h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) + o(\varepsilon)$. Therefore, as $\frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) = 1$, overbidding $\tilde{x} + \varepsilon$ provides a higher payoff to bidder i. The best reply of bidder j is either to overbid $\tilde{z}$, if $v_j - \tilde{z} + h(\tilde{z}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon)$, or to underbid $\tilde{y}$.

[9] If $v_1 = v_2 = v$, both bidders have the same maximum bid $\bar{x}$. The unique possible symmetric bidding equilibrium in pure strategies, $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}$, is the highest $\tilde{x}$ such that $\tilde{x} \leq \bar{x}$ and $\frac{\partial h}{\partial x_i}(\bar{x}, \bar{x}) = 1$.

Excluding the particular case $\frac{\partial h}{\partial x_j}(x_j, \tilde{x}+\varepsilon) = 1$ for all $x_j < \tilde{x}$, there is at least a value $\tilde{y}$ such that $\frac{\partial h}{\partial x_j}(\tilde{y}, \tilde{x}+\varepsilon) < 1$ is satisfied. Then bidder j underbids the smallest possible $\tilde{y}$.[10] Therefore, ruling out the particular case $\frac{\partial h}{\partial x_i}(x_i, \tilde{y}) = 1$ for all $x_i \in (\tilde{y}, \tilde{x}+\varepsilon]$, there is at least a value $\tilde{y}+\kappa$ with $\kappa > 0$ such that bidder i underbids. Bidder j will then deviate by proposing again $\tilde{x}$ if v_2 is sufficiently high. Therefore, this is unstable and there is no Nash equilibrium in pure strategies. This result of the potential non-existence of bidding equilibrium in pure strategies can be extended to heterogeneous externality functions with a similar reasoning.

The existence of a bidding equilibrium in pure strategies is not guaranteed and is fundamentally contingent on a specific and favorable shape of the externality functions. Therefore, we are looking for the existence of a Nash equilibrium in mixed strategies. In the following we denote $F_i(x) \equiv \mathbb{P}(X_i \leq x)$ the cumulative distribution functions such that bidder i decides to submit a bid lower than x. $F_1, ..., F_n$ can be interpreted as the bidding (mixed) strategies where the support is $\mathbb{R}_+$. Whatever the outcome of the auction, once bidder i computes her expected utility, she takes the bids paid by all bidders into account, including her own. Remark that we do not know if the cumulative distributions $F_1, ..., F_n$ admit density functions. Therefore we use the Stieltjes integral for the expected utility and in the following to determine the existence of a mixed strategy Nash equilibrium.[11] Moreover, the Stieljes integral exists if the cumulative distribution F_i is a function of bounded variation[12] and then discontinuous in at most a countable set of points. It follows that the cumulative distributions might not have density functions, but can admit atoms and a finite number of discontinuous points as in the standard case with no externalities (Baye et al., 1996).

The next Lemma shows that a result determined by Siegel (2009) about atoms also holds in the all-pay auction with general price externalities.[13]

Lemma 1. *Assume that at least two bidders have an atom at x. Then all bidders with an atom at x lose with probability 1 by submitting a bid x.*

Proof. Consider $\mathcal{T}$, $|\mathcal{T}| \geq 2$, the set of bidders who have an atom at x, which is to submit a bid x with strictly positive probability. Let us show the event "a winning tie occurs at x when all bidders in $\mathcal{T}$ submit a bid x" has a zero probability.

Assume this event has a strictly positive probability. The object is allocated among the $|\mathcal{T}|$ winners, such as every $i \in \mathcal{T}$ gets her value divided by the number of winners, i.e., $\frac{v_i}{|\mathcal{T}|}$. Bidder i, submitting a bid x with a strictly positive probability, must have a positive payoff, $\frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i}) \geq 0$ for a given vector of the other bids $\tilde{x}_{-i}$. Then there exists an $\varepsilon > 0$ such

[10] Indeed, $-\tilde{y} + h(\tilde{y}, \tilde{x}+\varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x}+\varepsilon)$.

[11] See (for example) Carothers (2000, Chap. 14) about the use of the Stieltjes integral because of non-existence of density functions.

[12] A cumulative distribution can be built from the difference between two bounded monotone functions and then be a function of bounded variation.

[13] Siegel (2009) investigates a large structure of contests, named *all-pay contests*, which generalizes the all-pay auctions to multiple objects, non-linear payoffs and heterogenous bidders. Yet it does not cover the all-pay auctions with price externalities depending on all bids. Our result corresponds to the Siegel's Tie Lemma for a single object.

as $v_i - (x+\varepsilon) + h_i(x+\varepsilon, \tilde{\boldsymbol{x}}_{-i}) \geq \frac{v_i}{|\mathcal{T}|} - x + h_i(x, \tilde{\boldsymbol{x}}_{-i})$, i.e., for which i can increase her probability of winning to 1 and her payoff by submitting a bid $x+\varepsilon$. Therefore, the event "a winning tie occurs at x when all bidders in $\mathcal{T}$ submit a bid x" has a zero probability to occur. This implies there is at least one bidder in $\mathcal{N} \setminus \mathcal{T}$, which submits a higher bid than x and wins the auction. Hence, all bidders with an atom at x will lose with probability 1 by submitting a bid equal to x.

■

The Lemma 1 implies that ties, in which bidders win with a strictly positive probability, are zero probability events. It follows that bidder i's expected utility is given by,[14]

$$\begin{aligned}\mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) =& \Pi_{j\neq i}F_j(x_i)\left(v_i + \mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \max_{j\neq i} X_j \leq x_i\right)\right) \\ &+ (1-\Pi_{j\neq i}F_j(x_i))\left(\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \exists j \neq i, X_j > x_i\right)\right) - x_i \text{ for all } i = 1, ..., n \\ =& \Pi_{j\neq i}F_j(x_i)v_i - x_i + \mathbb{E}h_i(x_i, \boldsymbol{X}_{-i}) \text{ for all } i = 1, ..., n\end{aligned}$$

with $\boldsymbol{X}_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. A potential bidder takes part in the auction if for some positive bids her expected utility is at least equal to the externality she benefits by bidding zero. Formally, a bidder takes part in the auction if

$$\exists x_i \text{ such that } \mathbb{E}U_i(x_i, \boldsymbol{X}_{-i}) \geq \mathbb{E}\left(h_i(0, \boldsymbol{X}_{-i})\right)$$

If the externality is linear, a closed from solution is straightforward to determine. Indeed, the expected payoff with no externality would only be affected by an affine transformation. As the result from Baye et al. (1996) is invariant to affine transformations of expected utility, the mixed strategies are also invariant. Unfortunately, it is not possible to determine an analytic solution without providing a particular shape of the externality function h_i. This is the consequence of the general mapping between x_i and $\boldsymbol{x}_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. However we are able to show the existence of a bidding equilibrium in mixed strategies.[15]

Proposition 1. *Given assumptions A1 − A4 a mixed strategy Nash equilibrium exists.*

Nonetheless, given that the solution is defined on a closed and convex set of continuous distribution functions, we are able to show its existence by using the Schauder's fixed-point theorem. Remark that we can consider to apply neither the Knaster-Tarski's fixed-point theorem nor the Brouwer's fixed-point theorem because of the infinite-dimensional set of distribution functions. Other applications of the Schauder's fixed-point theorem are provided by Stokey et al.

[14] Remark that $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \max_{j\neq i} X_j \leq x_i\right) = \begin{cases} \frac{1}{\Pi_{j\neq i}F_j(x_i)} \int_{[0,x_i]^n} h_i(x_i, \boldsymbol{x}_{-i})\Pi_{j\neq i}dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ and $\mathbb{E}\left(h_i(x_i, \boldsymbol{X}_{-i}) \mid \exists j \neq i, X_j > x_i\right) = \begin{cases} \frac{1}{1-\Pi_{j\neq i}F_j(x_i)} \int_{([0,x_i]^n)^c} h_i(x_i, \boldsymbol{x}_{-i})\Pi_{j\neq i}dF_j(x_i) \text{ if } x_i > 0 \\ 0 \text{ otherwise} \end{cases}$ with $([0,x_i]^n)^c$ the complement of $[0,x_i]^n$.

[15] Some existence theorems in discontinuous games, such as in Simon and Zame (1990) and more recently in Bich and Laraki (2017), could also be applied to establish our result. However, it would require a substantially more laborious work. It is also interesting to remark that some existence results, which can be applied to the all-pay auction, cannot be fulfilled here either because of the heterogeneous values, see for example Becker and Damianov (2006), or the price externality.

(1989) for overlapping-generations models, Fudenberg and Tirole (1991) for mixed strategy Nash equilibria with uncountable actions sets, Amir (1996), Curtat (1996) and Balbus et al. (2015) for Markov equilibrium in stochastic intergenerational games, Anderson et al. (1998) for logit equilibria in all-pay auctions and Fey (2008) for pure strategy Bayesian-Nash equilibria in rent-seeking contests.

Proof of Proposition 1. As in Anderson et al. (1998), our proof to apply the Schauder's fixed-point theorem consists in two steps[16]: to establish a continuous mapping and a set of functions uniformly bounded and equicontinuous.

Let us consider two bidders, i and j. Notice that if the bidder i has the highest maximum bid, denoted $\bar{x}_i$, he obtains with probability 1 a payoff $v_i - \bar{x}_i + h_i(\bar{x}_i, x_j)$. Given Assumption A4, $\dfrac{\partial h_i}{\partial x_i}(x_i, x_j) \leq 1$, a decrease of $\bar{x}_i$ to the maximum of her rival, denoted $\bar{x}_j$, will increase her payoff to $v_i - \bar{x}_j + h_i(\bar{x}_j, x_j)$ without altering her probability of winning. Following the standard reasoning for multiple bidders developed by Baye et al. (1996), all active bidders must have the same maximum bid $\bar{x}$.[17]

We investigate the indifference principle, which is the expected utility is constant for any pure strategy profile of the other bidders with positive probability. Let us denote $G_i := \prod_{j=1, j \neq i}^{n} F_j$ and the vector $\boldsymbol{y}_{-i} = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$. Given Assumption A2 and the use of the Stieljes integral, the indifference principle leads to the derivative:

$$G_i'(x) = \frac{1}{v_i} - \frac{1}{v_i} \int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{2}$$

In the following, we show that equation (2) is well defined. Notice that the right side of equation (2) involves G_i to be a continuous distribution function. We denote $\boldsymbol{G}(x) = (G_1(x), ..., G_n(x))$ the vector of mixed strategies and $\mathcal{D}_i = \{G_i \in \mathcal{C}([0, \bar{x}]), ||G_i|| \leq 1\}$ the set of distribution functions continuous on $[0, \bar{x}]$, with $||.||$ the supremum norm, and $\mathcal{C}([0, \bar{x}])$ the set of continuous functions on $[0, \bar{x}]$. It follows the set $\boldsymbol{\mathcal{D}} \equiv \mathcal{D}_1 \times ... \times \mathcal{D}_n$ with the norm $||\boldsymbol{G}||_\infty = \max_{i=1,...,n} ||G_i||$. Therefore let us consider $T : \boldsymbol{\mathcal{D}} \longrightarrow \boldsymbol{\mathcal{D}}$ to be the mapping $\boldsymbol{G}(x) \longmapsto \boldsymbol{TG}(x)$, with $\boldsymbol{TG}(x) = (TG_1(x), ..., TG_n(x))$. Then T maps every element G_i from $\mathcal{D}_i$ to $\mathcal{D}_i$, which given the integration of equation (2), is a function characterized by

$$TG_i(x) \equiv \lambda_i x - \lambda_i \int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \text{ for all } i = 1, ..., n. \tag{3}$$

with $\lambda_i = \frac{1}{v_i}$. Remark that the set $\boldsymbol{\mathcal{D}}$, which includes all the continuous distribution functions, is closed and convex but also infinite-dimensional. The Schauder's fixed-point theorem is then required to prove that $\boldsymbol{G}(x)$ is a fixed-point of the operator T defined by equation (3):

Theorem 1 (Schauder, 1930)**.** *If $\boldsymbol{\mathcal{D}}$ is a closed convex subset of a normed space and $\boldsymbol{\mathcal{E}}$ a relatively compact subset of $\boldsymbol{\mathcal{D}}$, then every continuous mapping of $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$ has a fixed-point.*

[16] However, we face an entire different problem than Anderson et al. (1998) as they show the existence of a logit equilibrium in all-pay auctions.

[17] The proof for n bidders and all possible rankings of valuations is similar to the one without externality in Baye et al. (1996).

Therefore, we show in the following that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact and that T is a continuous operator from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.

Step 1. *$\boldsymbol{\mathcal{E}}$ is relatively compact.*

We use the Arzelà-Ascoli's theorem to characterize the relative compactness in the space of continuous functions $\mathcal{C}([0, \bar{x}])$.

Theorem 2 (Arzelà-Ascoli, 1895)**.** *A set of functions in $\mathcal{C}([0, \bar{x}])$, with the supremum norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0, \bar{x}]$.*

Thus, to establish that $\boldsymbol{\mathcal{E}} \equiv \{\boldsymbol{TG} \mid \boldsymbol{G} \in \boldsymbol{\mathcal{D}}\}$ is relatively compact, we prove that $\boldsymbol{\mathcal{E}}$ is uniformly bounded and equicontinuous on $[0, \bar{x}]$.

First let us show that $\boldsymbol{\mathcal{E}}$ is uniformly bounded. Assumption $A4$ implies that

$$\int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, \boldsymbol{y}_{-i}) \Pi_{j \neq i} dF_j(y_j) \leq 1.$$

$TG_i(x)$ is thus increasing and $|TG_i(x)| \leq TG_i(\bar{x}) = 1$, for all $x \in [0, \bar{x}]$, $G_i \in \mathcal{D}_i$, $i = 1, ..., n$. Therefore, $\boldsymbol{TG}$ is uniformly bounded for all $\boldsymbol{G} \in \boldsymbol{\mathcal{D}}$.

Second, let us now prove that $\boldsymbol{TG}$ is equicontinuous $\forall \boldsymbol{G} \in \boldsymbol{\mathcal{D}}$: $\forall \varepsilon, \exists \eta : |TG_i(x_1) - TG_i(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta$, $\forall G_i \in \mathcal{D}_i$ and $i = 1, ..., n$. To show it, notice that the function h_i is continuous and bounded on the compact $[0, \bar{x}]$. We can then compute,

$$\begin{aligned}
|TG_i(x_1) - TG_i(x_2)| &= \left| \lambda_i (x_1 - x_2) - \lambda_i \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})] \Pi_{j \neq i} dF_j(y_j) \right| \\
&\leq \lambda_i \left[|x_1 - x_2| + \left| \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})] \Pi_{j \neq i} dF_j(y_j) \right| \right] \\
&\leq \lambda_i |x_1 - x_2| \left[1 + \frac{|\sup_{\boldsymbol{y}_{-i} \in [0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})]|}{|x_1 - x_2|} \right] \\
&< \lambda_i \eta \left[1 + \frac{|\sup_{\boldsymbol{y}_{-i} \in [0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})]|}{|x_1 - x_2|} \right].
\end{aligned}$$

Thus, $|TG_i(x_1) - TG_i(x_2)| < \varepsilon$ for $\eta = \varepsilon \min\limits_{i=1,...,n} \dfrac{|x_1 - x_2|}{\lambda_i(|x_1 - x_2| + \kappa_i)}$ for all $G_i \in \mathcal{D}_i$ and $i = 1, ..., n$, with $\kappa_i \equiv |\sup_{\boldsymbol{y}_{-i} \in [0,\bar{x}]^{n-1}} [h_i(x_1, \boldsymbol{y}_{-i}) - h_i(x_2, \boldsymbol{y}_{-i})]|$.

Step 2. *T is a continuous mapping from $\boldsymbol{\mathcal{D}}$ to $\boldsymbol{\mathcal{E}}$.*

To establish T is a continuous mapping, we define $\hat{G}_i(x) = \Pi_{j \neq i} \hat{F}_j(x)$ and $\tilde{G}_i(x) = \Pi_{j \neq i} \tilde{F}_j(x)$ such as $\hat{G}_i(x) = \tilde{G}_i(x) + k_i(x)$ with $|k_i(x)| < \eta$ for all $x \in [0, \bar{x}]$, $i = 1, ..., n$, and show that for all $\hat{\boldsymbol{G}}, \tilde{\boldsymbol{G}} \in \boldsymbol{\mathcal{D}}$ and for all $\varepsilon > 0$, there is a $\eta > 0$ such that $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty < \varepsilon$ when $||\hat{\boldsymbol{G}} - \tilde{\boldsymbol{G}}||_\infty < \eta$. We can compute,

$$|T\hat{G}_i(x) - T\tilde{G}_i(x)| = \left| -\lambda_i \left(\int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} h_i(x, \boldsymbol{y}_{-i}) \Pi_{j\neq i} d\tilde{F}_j(y_j) \right) \right|$$
$$\leq \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) \left| \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\hat{F}_j(y_j) - \int_{[0,\bar{x}]^{n-1}} \Pi_{j\neq i} d\tilde{F}_j(y_j) \right|$$
$$= \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) |k_i(\bar{x})|$$
$$< \lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}) \eta.$$

As h_i is a continuous function in all arguments, bounded by $\sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i})$, the second line follows. The transition from the second to the third line is from the independence of the density functions and $\hat{G}_i(x) - \tilde{G}_i(x) = k_i(x)$. Therefore, $||\boldsymbol{T}\hat{\boldsymbol{G}}(x) - \boldsymbol{T}\tilde{\boldsymbol{G}}(x)||_\infty$ is inferior to $\varepsilon > 0$ when $\eta = \min_{i=1,\dots,n} \frac{\varepsilon}{\lambda_i \sup_{\boldsymbol{y}_{-i}\in[0,\bar{x}]^{n-1}} h_i(\bar{x}, \boldsymbol{y}_{-i}))}$ for all $x \in [0, \bar{x}]$.

■

4 Concluding Remarks

We show the existence of a mixed strategy Nash equilibrium for all-pay auctions with general price externalities. The proof relies on the Schauder's fixed-point theorem, which does not require a finite-dimensional set of the distribution functions. Our result can be useful for many economic applications such as charity mechanisms and contests in which relative performance matters.

Unfortunately, there is no closed form solution to this problem without providing a specific form of the externality functions. A numerical approach combined with a lab experiment would be useful to determine how non-linearity can affect the revenue performance of the all-pay auction. Uniqueness is also an important investigation for future research. The two-bidder case leads to a first-order condition which can be identified as a Fredholm equation. Kanwal (1971) provides a sufficient condition for uniqueness, $\sup_{x\in[0,\bar{x}]} \int_0^{\bar{x}} \left| \frac{\partial h_i}{\partial x_i}(x_i, x_j) \right| dx_j < 1$, which is quite restrictive and seems unrelevant for an economic analysis.

To sum up, natural follow-up questions concern the extent of uniqueness and numerical approach to determine the properties of the equilibrium. These could have many interests for applications in fundraising mechanisms and race contests.

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