Existence of Equilibrium in All-Pay Auctions with Price Externalities

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Abstract

We investigate all-pay auctions with general price externalities and complete information. We show the existence of a mixed strategy Nash equilibrium by using Schauder’s fixed-point theorem. The Brouwer’s fixed-point cannot be applied because of non compactness of the set of distribution functions. Our findings are applicable to future works on contests and charity auctions.

Keywords: All-pay auction, price externalities

JEL Classification: D44, D62, C72

1 Introduction

All-pay auctions are used both as an auction device and a contest. In this game, all bidders have to pay their bids and the one who submitted the highest bid wins. Fundraising mechanisms and race competitions are two of the many applications. Competitors on a race care about the recognition they could get from their participation. Therefore, a loser is better off with an effort closer to the winner’s effort and a winner with an effort further to the highest loser’s effort. In a charity auction, bidders usually care about the charity purpose and benefit from the total amount raised during the auction. In both situations, participants benefit from a price externality, either dependent or independent of the winner identity.1

The most famous result in this literature is that all-pay auctions are optimal mechanisms at raising money for charity (Goeree et al. (2005) and Engers and McManus (2007)). However,

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1Notice there is also a large literature about auctions with allocative externalities initiated by Jehiel and Moldovannu (1996) with complete information. The key in this literature is the winner identity and not, as here, the money/efforts spent. See Klose and Kovenock (2015) for an analysis of the all-pay auction with allocative externalities.
this is not confirmed on the field. Carpenter et al. (2008) and Onderstal et al. (2013) observe a low participation in field experiments and determine other better fundraising mechanisms. That might be due to the assumption of linear externality that participants benefit.\footnote{Among the other possible explanations, Carpenter et al. (2010) suggest the unfamiliarity of the participants to the mechanism and endogenous participation (which is not considered here) and Bos (2016) heterogeneity among bidders.} Indeed, all theoretical results about charity auctions rely on this linearity assumption. Non linear externalities could lead to other optimal mechanisms and therefore affect the bidders’ participation at the equilibrium. In this paper, we propose to investigate the all-pay auction with complete information without specifying the shape of the externality functions. Complete information is not as usual in auction theory as in the contest literature. However, there are recent works about auctions with externalities which investigate a setting with complete information.\footnote{Among them, Jehiel and Moldovanu (1996) consider auctions with allocative externalities, Ettinger (2010) investigates first and second-price winner-pay auctions with price externalities, Bos (2016) compares the performance of all-pay auctions and winner-pay auctions for charity purpose and Damianov and Peeters (2018) compare the lowest-price all-pay auctions with other fundraising mechanisms.}

Our analysis does not make any assumptions on the analytic form of the externality function, and consider that bidders take in account a positive externality from their own bid and either a positive or a negative externality from their rivals’ bids. Thus, the externality function might also reflect a statistical link between the bids of a bidder and her competitors. These are relevant with economic applications, and more specifically with charity auctions and race competitions. We establish the existence of a Nash equilibrium with mixed strategies, defined on a closed, convex and infinite dimension set of continuous distribution functions. Therefore we use the Schauder’s fixed-point theorem to determine this result despite the non compactness of this set. That might be useful for future research on fundraising mechanisms.

The remainder of this paper is structured as follow. Section 2 introduces the formal setting and properties of general price externalities. In Section 3 we discuss the non-existence of a pure strategy Nash equilibrium and show the existence of a mixed strategy Nash equilibrium. We conclude in Section 4. In the following we refer to the players as bidders, keeping the auction terminology.

2 The Model

Suppose \( n \) risk-neutral bidders submits their bids for an indivisible object (or prize) which is allocated to the highest bidder. Bidder \( i \)'s value is given by \( v_i \) such that \( v_1 \geq v_2 \geq \ldots \geq v_n \geq 0 \). Although valuations are common knowledge among the potential bidders, the seller has no information about them. All bidders have to pay their bids. Therefore, either bidder \( i \) wins the auction with a bid \( x_i \), and obtains a payoff \( v_i - x_i \), or she loses and obtains \( -x_i \).\footnote{If more than one bidder submit the same winning bid, they win with equal probability.} Moreover, the money raised from each potential bidder potentially impacts the utility of each other: each bidder benefits from her own participation and either benefits or suffers from her rivals’ bids.
Thus, the bidder’s utility function includes a price externality depending on all bids paid. That could be a function of the sum of the bids $\sum_{i=1}^{n} x_i$, the revenue raised, as in charity auctions, and thus independent of the winner’s identity. That could also be a function of the difference between bids contingent to the winner’s identity. Indeed in contests, participants might get a reward not only from winning but also from her relative performance compared to others. Accordingly we consider an externality function dependent on all bids, $h_i : \mathbb{R}_+^n \to \mathbb{R}_+$. It follows that bidder $i$’s utility is given by:

$$U_i(x_i, x_{-i}) = \begin{cases} 
    v_i - x_i + h_i(x_i, x_{-i}) & \text{if } i \text{ is the only winner} \\
    \frac{\alpha}{\sum_{j=1}^{n}} x_i + h_i(x_i, x_{-i}) & \text{if } i \text{ is one of the } k \text{ winners} \\
    -x_i + h_i(x_i, x_{-i}) & \text{otherwise}
\end{cases} \quad (1)$$

with $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. The linear case can lead to $\alpha \sum_{i=1}^{n} x_i$, with $\alpha$ a positive number, which is the usual form of the externality function in the charity auction literature and other works about auctions with identity-independent price externalities. For the analysis we make the following assumptions, relevant with economic applications.

**Assumption 1 (A1).** If all bidders submit a zero bid, none benefit from the externality: $h_i(0, ..., 0) = 0$.

**Assumption 2 (A2).** The externality function is differentiable in all its arguments.

As stated by Assumption A1, it is relevant that non active participation of all bidders does not induce any benefit, while A2 is a technical assumption.

**Assumption 3 (A3).** Bidder $i$ benefits from an increasing externality in her own bid.

A higher bid from $i$ participates to the success of the auction and then improves her payoff, which leads to $\frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) \geq 0$. Despite bidder $i$ benefits positively from her own bid, it is not necessarily the case from her rivals’ bids. This is contingent to the shape of $h_i$ and therefore how bidder $i$ perceives her competitors’ bids and the statistical link between all bids.

**Assumption 4 (A4).** Bidder $i$’s payoff is decreasing with her payment $x_i$ in the auction.

Assumption A4 means bidders are always better off by paying a lower price and leads to $\frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) \leq 1$. In all paper about all-pay auctions with price externalities, it is either implicitly or explicitly assumed that the payoff cannot increase with the bidder payment, which means that $\frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) < 1$. Applied to charity auctions, this assumption displays the limit of the bidders’ altruism. Otherwise bidders could be indifferent between giving and keeping money for their personnel use. In a contest, such as a race, a competitor gets a higher (lower) payoff from a la Andreoni (1989), making bidders more sensitive to their own bid for cognitive/psychological reasons and thus benefiting differently from their own bids and their rivals’ bids. Therefore, bidder $i$ benefits from the externality $\alpha x_i + \beta \sum_{j=1,j\neq i}^{n} x_j$ with $1 > \alpha \geq \beta > 0$. 

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5Bidder $i$ might be one of the $k$ winners, all submitting the same highest bid, such that $k = \#\{j \mid j = \arg\max\{x_i, i \in \{1, \ldots, n\}\}$.

6See Goeree et al. (2005), Engers and McManus (2007) and Bos (2016).

7See for example Ettinger (2010).

8Goeree et al. (2005) and Bos (2016) consider the linear case $\alpha \sum_{i=1}^{n} x_i$, and assume that $\alpha$ is strictly inferior to 1. Engers and McManus (2007) add a warm glow a la Andreoni (1989), making bidders more sensitive to their own bid for cognitive/psychological reasons and thus benefiting differently from their own bids and their rivals’ bids. Therefore, bidder $i$ benefits from the externality $\alpha x_i + \beta \sum_{j=1,j\neq i}^{n} x_j$ with $1 > \alpha \geq \beta > 0$. 

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a better (worst) relative performance, \( \alpha_i(x_i - \max_{j \neq i} x_j) \), due to reputation or recognition. Yet the marginal cost has to be higher than the marginal benefit from her own performance, otherwise her effort might go to infinity. That requires again \( \alpha_i < 1 \), which is here \( \frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) < 1 \).

Our investigation is more general. We also consider the limit case such that bidders could be fully altruistic in a charity auction and the marginal cost of the winner in a race could be equal to her marginal benefit.

### 3 Existence of an Equilibrium

In this section, we show the existence of a Nash equilibrium in mixed strategies. To understand better the extent of this result, we first discuss how the (non) existence of a Nash equilibrium in pure strategies is contingent to the shape of the externality functions.

It is a well known result there is no bidding equilibrium in pure strategies in all-pay auctions without price externality (see Hillman and Riley (1989) and Baye et al. (1996)). Unsurprisingly, that is also the case for the class of externality functions such that \( \frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) < 1 \) for all \( i = 1, \ldots, n \).

We only provide a sketch of the proof with two-bidder to make the argument easier to follow. Let us assume that \( x_i \geq x_j \), then two cases may arise. First if bidder \( j \) can overbid, her profitable deviation is \( x_i + \varepsilon \) for \( \varepsilon > 0 \), such that \( v_j - (x_i + \varepsilon) + h_j(x_i, x_i + \varepsilon) \geq -x_j + h_j(x_i, x_j) \). In this case, \( x_i \geq x_j \) is not possible. Second if \( j \) cannot overbid, her profitable deviation is to offer zero since \( h_j(0, x_i) > -x_j + h_j(x_j, x_i) \) given \( \frac{\partial h_j}{\partial x_j}(x_j, x_i) < 1 \). Therefore, \( i \)'s profitable deviation is to offer \( \varepsilon > 0 \). As a result, this is unstable and there is no pure strategy Nash equilibrium.

There are also many situations in our general case described by assumption A4, \( \frac{\partial h_i}{\partial x_i}(x_i, x_{-i}) \leq 1 \) for all \( i = 1, \ldots, n \), in which there is no Nash equilibrium in pure strategies. To make the statement more straightforward to follow, we focus again on the two-bidder case.

Let suppose both bidders maximize \( v_i - x_i + h_i(x_i, x_j) \) for \( i = 1, 2, i \neq j \). Then they choose \((\hat{x}_1, \hat{x}_2)\) such that

\[
\frac{\partial h_1}{\partial x_1}(\hat{x}_1, \hat{x}_2) = \frac{\partial h_2}{\partial x_2}(\hat{x}_2, \hat{x}_1) = 1.
\]

If both bidders benefit from the same externality functions, \( h_1 = h_2 \equiv h \), they bid \( \hat{x}_1 = \hat{x}_2 = \hat{x} \) and get a payoff \( \frac{v_i}{2} - \hat{x} + h(\hat{x}, \hat{x}) \) for \( i = 1, 2 \). Let us now consider \( v_1 > v_2 \) such that bidders do not have the same maximum bid. In this case, bidder \( i \) is always better off by overbidding \( x_i = \tilde{x} + \varepsilon \) for \( \varepsilon > 0 \) and \( \tilde{x}_2 = \tilde{x} \) cannot be an equilibrium.\(^9\) Using the Taylor’s theorem at the point \( \tilde{x} \) such that \( h(\tilde{x} + \varepsilon, \hat{x}) = h(\hat{x}, \hat{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\hat{x}, \hat{x}) + o(\varepsilon) \) with \( \lim_{\varepsilon \to 0} o(\varepsilon) = 0 \), a bid \( \tilde{x} + \varepsilon \) leads to the payoff \( v_i - \tilde{x} - \varepsilon + h(\tilde{x}, \tilde{x}) + \varepsilon \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) + o(\varepsilon) \). Therefore, as \( \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) = 1 \), overbidding \( \tilde{x} + \varepsilon \) provides a higher payoff to bidder \( i \). The best reply of bidder \( j \) is either to overbid \( \hat{z} \), if \( v_j - \hat{z} + h(\hat{z}, \hat{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \hat{x} + \varepsilon) \), or to underbid \( \hat{y} \).

\(^9\)If \( v_1 = v_2 = v \), both bidders have the same maximum bid \( \hat{x} \). The unique possible symmetric bidding equilibrium in pure strategies, \( \tilde{x}_1 = \tilde{x}_2 = \tilde{x} \), is the highest \( \tilde{x} \) such that \( \tilde{x} \leq \hat{x} \) and \( \frac{\partial h}{\partial x_i}(\tilde{x}, \tilde{x}) = 1 \).
Excluding the particular case \( \frac{\partial h}{\partial x_j}(x_j, \tilde{x} + \varepsilon) = 1 \) for all \( x_j < \tilde{x} \), there is at least a value \( \tilde{y} \) such that \( \frac{\partial h}{\partial x_j}(\tilde{y}, \tilde{x} + \varepsilon) < 1 \) is satisfied. Then bidder \( j \) underbids the smallest possible \( \tilde{y} \). Therefore, ruling out the particular case \( \frac{\partial h}{\partial x_i}(x_i, \tilde{y}) = 1 \) for all \( x_i \in (\tilde{y}, \tilde{x} + \varepsilon] \), there is at least a value \( \tilde{y} + \kappa \) with \( \kappa > 0 \) such that bidder \( i \) underbids. Bidder \( j \) will then deviate by proposing again \( \tilde{x} \) if \( \nu_2 \) is sufficiently high. Therefore, this is unstable and there is no Nash equilibrium in pure strategies. This result of the potential non-existence of bidding equilibrium in pure strategies can be extended to heterogeneous externality functions with a similar reasoning.

The existence of a bidding equilibrium in pure strategies is not guaranteed and is fundamentally contingent on a specific and favorable shape of the externality functions. Therefore, we are looking for the existence of Nash equilibria in mixed strategies. In the following we denote \( F_j(x) \equiv P(X_i \leq x) \) the cumulative distribution functions such that bidder \( i \) decides to submit a bid lower than \( x \). \( F_1, ..., F_n \) can be interpreted as the bidding (mixed) strategies where the support is \( \mathbb{R}_+ \). Whatever the outcome of the auction, once bidder \( i \) computes her expected utility, she takes the bids paid by all bidders into account, including her own. Bidder \( i \)'s expected utility is given by\(^{11}\),

\[
\begin{align*}
EU_i(x_i, X_{-i}) &= \Pi_{j \neq i} F_j(x_i) \left( v_i + E \left( h_i(x_i, X_{-i}) \max_{j \neq i} X_j \leq x_i \right) \right) \\
&\quad + (1 - \Pi_{j \neq i} F_j(x_i)) \left( E \left( h_i(x_i, X_{-i}) \exists j \neq i, X_j > x_i \right) \right) - x_i \text{ for all } i = 1, ..., n \\
&= \Pi_{j \neq i} F_j(x_i) v_i - x_i + E h_i(x_i, X_{-i}) \text{ for all } i = 1, ..., n
\end{align*}
\]

with \( X_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n) \). A potential bidder takes part in the auction if for some positive bids her expected utility is at least equal to the externality she benefits by bidding zero. Formally, a bidder takes part in the auction if

\[
\exists x_i \text{ such that } EU_i(x_i, X_{-i}) \geq E \left( h_i(0, X_{-i}) \right)
\]

If the externality is linear, a closed from solution is straightforward to determine. The expected payoff with no externality would only be affected by an affine transformation. As the result from Baye et al. (1996) is invariant to affine transformations of expected utility, the mixed strategies are also invariant. Unfortunately, it is not possible to determine an analytic solution without providing a particular shape of the externality function \( h_i \). This is the consequence of the general mapping between \( x_i \) and \( X_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \). However we are able to show the existence of a bidding equilibrium in mixed strategies.\(^{12}\)

\(^{10}\)Indeed, \(-\tilde{y} + h(\tilde{y}, \tilde{x} + \varepsilon) > -\tilde{x} + h(\tilde{x}, \tilde{x} + \varepsilon)\).

\(^{11}\)Remark that \( E \left( h_i(x_i, X_{-i}) \max_{j \neq i} X_j \leq x_i \right) = \left\{ \begin{array}{ll} \frac{1}{\Pi_{j \neq i} F_j(x_i)} \int_{[0,x_i]^n} h_i(x_i, X_{-i}) \Pi_{j \neq i} dF_j(x_i) & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{array} \right. \)

and \( E \left( h_i(x_i, X_{-i}) \exists j \neq i, X_j > x_i \right) = \left\{ \begin{array}{ll} \frac{1}{1 - \Pi_{j \neq i} F_j(x_i)} \int_{(0,x_i]^n} h_i(x_i, X_{-i}) \Pi_{j \neq i} dF_j(x_i) & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{array} \right. \) with \((0, x_i]^n\) the complement of \([0, x_i]^n\).

\(^{12}\)Some existence theorems in discontinuous games, such as in Simon and Zame (1990) and more recently
Proposition 1. Given assumptions A1 – A4 a mixed strategy Nash equilibrium exists.

Nonetheless, given that the solution is defined on a closed and convex set of continuous distribution functions, we are able to show its existence by using the Schauder’s fixed-point theorem. Remark that we cannot consider to apply neither the Knaster-Tarski’s fixed-point theorem nor the Brouwer’s fixed-point theorem because of infinite dimension and therefore non compactness.

Proof of Proposition 1. As in Anderson et al. (1998), our proof to apply the Schauder’s fixed-point theorem consists in two steps: to establish a continuous mapping and a set of functions uniformly bounded and equicontinuous.\(^\text{13}\)

Notice that if the bidder \(i\) has a higher maximum bid \(\bar{x}_i\) than the others, a decrease of \(\bar{x}_i\) to the maximum of her rivals will increase her payoff without altering her probability of winning. Following this standard reasoning for multiple bidders with equal maximum bids higher than the other bidders, all bidders must have the same maximum bid \(\bar{x}\). The set of equilibria in mixed strategies is completely characterized by the Nash equilibria where only pure strategies, which are better responses to the other strategies, are played with a strictly positive probability. As the expected utility is constant at the equilibrium, the first order condition leads to

\[
g_i(x) = \frac{1}{v_i} - \frac{1}{v_i} \int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i(x, y_{-i})}{\partial x_i} \Pi_{j \neq i} f_j(y_j) dy_{-i} \quad \text{for all } i = 1, \ldots, n. \tag{2}
\]

with \(dy_{-i} = dy_1 \times \ldots \times dy_{i-1} \times dy_{i+1} \ldots \times dy_n\). Remark that assumption A4 implies that the function \(g_i\) and \(f_i\) are density functions such that \(G_i(x) = \prod_{j=1, j \neq i}^{n} F_j(x)\) and \(G_i'(x) = g_i(x) = \sum_{j \neq i} \prod_{k \neq i, j} F_k(x) f_j(x)\). In the following we denote \(\lambda_i = \frac{1}{v_i}\) and \(G(x) = (G_1(x), \ldots, G_n(x))\) the vector of mixed strategies. Let us now consider \(T\) an operator such that \(T: G(x) \mapsto TG(x)\), which given the integration of equation (2) is characterized by

\[
TG_i(x) = \lambda_i x - \lambda_i \int_{[0,\bar{x}]^{n-1}} h_i(x, y_{-i}) \Pi_{j \neq i} f_j(y_j) dy_{-i} \quad \text{for all } i = 1, \ldots, n. \tag{3}
\]

We would like to show that a mixed strategy Nash equilibrium exists, and therefore \(TG_i(x) = G_i(x)\) for all \(i = 1, \ldots, n\) and \(x \in [0,\bar{x}]\) which implies \(TF_i(x) = F_i(x)\) for all \(i = 1, \ldots, n\) and \(x \in [0,\bar{x}]\).

Notice that even if the density functions would not be continuous, equation (3) involves \(G_i\) and then \(F_i\) to be continuous distribution functions. Therefore, we denote \(D_i = \{G_i \in C([0,\bar{x}]), ||G_i|| \leq 1\}\) the set of distribution functions continuous on \([0,\bar{x}]\), with ||.|| the supremum norm and \(C([0,\bar{x}])\) the set of continuous functions on \([0,\bar{x}]\). It follows that the set \(D = D_1 \times \ldots \times D_n\), with the norm \(||G||_n = \max_{i=1, \ldots, n} ||G_i||\), which includes all the continuous distribution

Bich and Laraki (2017), could also be applied to establish our result. However, it would require a substantial more laborious work. It is also interesting to remark that some existence results, which can be applied to the all-pay auction, cannot be fulfilled here either because of the heterogeneous values, see for example Becker and Damianov (2006), or the price externality.

\(^{13}\)However, we face an entirely different problem than Anderson et al. (1998) as they show the existence of a logit equilibrium in all-pay auctions.
functions, is closed and convex but not compact (as it is an infinite dimensional set). The Schauder’s fixed-point theorem is then required to prove that $G(x)$ is a fixed-point of the operator $T$ defined by equation (3):

**Theorem 1** (Schauder, 1930). *If $\mathcal{D}$ is a closed convex subset of a normed space and $\mathcal{E}$ a relatively compact subset of $\mathcal{D}$, then every continuous mapping of $\mathcal{D}$ to $\mathcal{E}$ has a fixed-point.*

Therefore, we show in the following that $\mathcal{E} \equiv \{TG \mid G \in \mathcal{D}\}$ is relatively compact and that $T$ is a continuous mapping from $\mathcal{D}$ to $\mathcal{E}$.

**Step 1.** $\mathcal{E}$ is relatively compact.

We use the Arzelà-Ascoli’s theorem to characterize the relative compactness in the space of continuous functions $\mathcal{C}([0, \bar{x}])$.

**Theorem 2** (Arzelà-Ascoli, 1895). *A set of functions in $\mathcal{C}([0, \bar{x}])$, with the supremum norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0, \bar{x}]$.***

Thus, to establish that $\mathcal{E} \equiv \{TG \mid G \in \mathcal{D}\}$ is relatively compact, we prove that $\mathcal{E}$ is uniformly bounded and equicontinuous on $[0, \bar{x}]$.

First let us show that $\mathcal{E}$ is uniformly bounded. Assumption A4 implies that

$$\int_{[0,\bar{x}]^{n-1}} \frac{\partial h_i}{\partial x}(x, y_{-i}) \Pi_{j \neq i} f_j(y_j) dy_{-i} \leq 1.$$ 

$TG_i(x)$ is thus increasing and $|TG_i(x)| \leq TG_i(\bar{x}) = 1$, for all $x \in [0, \bar{x}]$, $G \in \mathcal{D}$, $i = 1, \ldots, n$. Therefore, $TG$ is uniformly bounded for all $G \in \mathcal{D}$.

Second, let us now prove that $TG$ is equicontinuous $\forall G \in \mathcal{D}$: $\forall \varepsilon, \exists \eta : |TG_i(x_1) - TG_i(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta$, $\forall G \in \mathcal{D}$ and $i = 1, \ldots, n$. To show it, notice that the function $h_i$ is continuous and bounded on the compact $[0, \bar{x}]$. We can then compute,

$$|TG_i(x_1) - TG_i(x_2)| = \lambda_i |x_1 - x_2| - \lambda_i \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, y_{-i}) - h_i(x_2, y_{-i})] \Pi_{j \neq i} f_j(y_j) dy_{-i}$$

$$\leq \lambda_i |x_1 - x_2| + \int_{[0,\bar{x}]^{n-1}} [h_i(x_1, y_{-i}) - h_i(x_2, y_{-i})] \Pi_{j \neq i} f_j(y_j) dy_{-i}$$

$$\leq \lambda_i |x_1 - x_2| \left[ 1 + \frac{\sup_{y_{-i} \in [0, \bar{x}]^{n-1}} [h_i(x_1, y_{-i}) - h_i(x_2, y_{-i})]}{|x_1 - x_2|} \right]$$

$$< \lambda_i \eta \left[ 1 + \frac{\sup_{y_{-i} \in [0, \bar{x}]^{n-1}} [h_i(x_1, y_{-i}) - h_i(x_2, y_{-i})]}{|x_1 - x_2|} \right].$$

Thus, $|TG_i(x_1) - TG_i(x_2)| < \varepsilon$ for $\eta = \varepsilon \min_{i=1, \ldots, n} \frac{|x_1 - x_2|}{\lambda_i (|x_1 - x_2| + \kappa_i)}$ for all $G \in \mathcal{D}$ and $i = 1, \ldots, n$, with $\kappa_i \equiv \sup_{y_{-i} \in [0, \bar{x}]^{n-1}} [h_i(x_1, y_{-i}) - h_i(x_2, y_{-i})]$.

**Step 2.** $T$ is a continuous mapping from $\mathcal{D}$ to $\mathcal{E}$. 

To establish $T$ is a continuous mapping, we define $\hat{G}_i(x) = \tilde{G}_i(x) + g_i(x)$ with $|g_i(x)| < \eta$ for all $x \in [0, \bar{x}]$, $i = 1, \ldots, n$, and show that for all $\hat{G}, \tilde{G} \in D$ and for all $\varepsilon > 0$, there is a $\eta > 0$ such that $||T\hat{G}(x) - T\tilde{G}(x)||_n < \varepsilon$ when $||\hat{G} - \tilde{G}||_n < \eta$. We can compute,

$$||T\hat{G}_i(x) - T\tilde{G}_i(x)|| = -\lambda_i \int_{[0,\bar{x}]^{n-1}} h_i(x, y_{-i}) \left( \Pi_{j\neq i} \hat{f}_j(y_j) - \Pi_{j\neq i} \tilde{f}_j(y_j) \right) dy_{-i}$$

$$\leq \lambda_i \sup_{y_{-i} \in [0,\bar{x}]^{n-1}} h_i(x, y_{-i}) \int_{[0,\bar{x}]^{n-1}} \left( \Pi_{j\neq i} \hat{f}_j(y_j) - \Pi_{j\neq i} \tilde{f}_j(y_j) \right) dy_{-i}$$

$$= \lambda_i \sup_{y_{-i} \in [0,\bar{x}]^{n-1}} h_i(x, y_{-i}) |g_i(x)|$$

$$< \lambda_i \sup_{y_{-i} \in [0,\bar{x}]^{n-1}} h_i(x, y_{-i}) \eta.$$

As $h_i$ is a continuous function in all arguments, bounded by $\sup_{y_{-i} \in [0,\bar{x}]^{n-1}} h_i(x, y_{-i})$, the second line follows. The transition from the second to the third line is from the independence of the density functions and $\hat{G}_i(x) - \tilde{G}_i(x) = g_i(x)$. Therefore, $||T\hat{G}(x) - T\tilde{G}(x)||_n$ is inferior to $\varepsilon > 0$ when $\eta = \min_{i=1,\ldots,n} \lambda_i \sup_{y_{-i} \in [0,\bar{x}]^{n-1}} h_i(x, y_{-i})$ for all $x \in [0, \bar{x}]$.

4 Concluding Remarks

We show the existence of a mixed strategy Nash equilibrium for all-pay auctions with general price externalities. The proof relies on the Schauder’s fixed-point theorem which does not require the compactness of the set of the continuous distribution functions. Our result can be useful for many economic applications such as charity mechanisms and contests in which relative performance matters.

Unfortunately, there is no closed form solution to this problem without providing a specific form of the externality functions. A numerical approach combined with a lab experiment would be useful to determine how non-linearity can affect the revenue performance of the all-pay auction. Uniqueness is also an important investigation for future research. The two-bidder case leads to a first-order condition which can be identified as a Fredholm equation. Kanwal (1971) provides a sufficient condition for uniqueness, $\sup_{x \in [0,\bar{x}]} \int_0^x |\frac{\partial h_i}{\partial x_i}(x, x_j)| dx_j < 1$, which is quite restrictive and seems irrelevant for an economic analysis.

To sum up, natural follow-up questions concern the extent of uniqueness and numerical approach to determine the properties of the equilibrium. These could have many interests for applications in fundraising mechanisms and race contests.

References


