

# All-Pay Auctions with Resale\*

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## Abstract

We study equilibria of first- and second-price all-pay auctions with resale when players' signals are affiliated and symmetrically distributed. We show that existence of resale possibilities introduces an endogenous element to players' valuations and creates a signaling incentive for players. We characterize symmetric bidding equilibria for both first- and second-price all-pay auctions with resale and provide sufficient conditions for existence of symmetric equilibria. Under our conditions we show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale. The initial seller could benefit from publicly disclosing his private information which is affiliated with players' signals.

*Keywords:* all-pay auctions, resale

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# 1 Introduction

This paper studies all-pay auctions with resale. Contrasting with standard auctions where only winners are required to make payments, all-pay auctions exhibit special characteristic of unconditional payment, that is, bidders always pay their bids regardless winning or losing. All-pay auctions or equivalent models have been widely used to model a variety of economic and social instances of conflict and competition such as lobbying, contests and tournaments, political campaigns, patent races, and so on.<sup>1</sup> Two formats of all-pay auctions are widely used in the existing literature: first-price all-pay auctions and second-price all-pay auctions.<sup>2</sup> They differ only in winning bidder's payment. The winning bidder pays his own bid in first-price all-pay auctions, and the highest losing bid in second-price all-pay auctions. In both auctions, all losing bidders pay their own bids. Hence, they are analogous to standard first-price and second-price sealed-bid auctions.

Using the general symmetric model, Krishna and Morgan (1997) study all-pay auctions by characterizing equilibrium strategies and comparing expected revenues resulting from both first-price and second-price all-pay auctions. However, there are many instances a static all-pay auction model could not account for. Participants in such instances often face after-market competitions. Several examples are in order.

*Patent Races.* Patent races are often formalized as all-pay auctions since resources devoted by competitors are irreversible, regardless winning or losing. Winners in patent races can either retain exclusive use of the innovations or license the innovations for use by other producers. When an innovation is to be patented, the winner who comes from a research institution is very likely to sell the patent to producers interested. On the demand side, an incumbent monopolist possessing related or substitutable technology has more incentive than

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<sup>1</sup>See, for example, Baye et al. (1993), Krishna and Morgan (1997), and Moldovanu and Sela (2001).

<sup>2</sup>Second-price all-pay auctions are better known as the war of attrition and are used to model conflicts among animals (Maynard Smith, 1982) and struggles for survival among firms (Fudenberg and Tirole, 1986).

a potential entrant to buy the patent.<sup>3</sup> Hence, very often there is a secondary market for patents. In a broader view, the resale of patented technology takes place all over the world. In particular, the transfer of patented technology from developed countries to developing ones promotes the economic performances of the latter. This could be evidenced by the development of many east Asian countries.<sup>4</sup>

*Lobbying.* Rent-seeking activities such as lobbying play an important role in the allocation of government contracts. Lobbyists make implicit payments to politicians through campaign contributions or other channels in order to influence political decisions. Lobbying is usually formalized as an all-pay auction since lobbyists' up-front payments are not refundable to those failing to win the prize. Again there are aftermarket competitions in lobbying. Successful lobbyists often use subcontracting to reduce their production costs that are, in particular, strictly convex.<sup>5</sup>

*Waiting-Line Competition.* The allocation procedures based on a first-come-first-served principle could be considered as waiting-line competitions. Examples include allocating tickets of sports or concerts, foods or other necessities with scarcity, university parking lots or day-care services, discounted commodities, and so on. These waiting-line allocation procedures are often formalized as all-pay auctions, since players are involved into costly competition in some non-price dimensions for a limited number of prizes. No matter winning or losing, players' effort is sunk. Although resale is usually not allowed, in practice, speculative behaviors are prevalent. For instance, people with lower time cost could wait in line and profit by reselling the objects to those with higher opportunity cost of time.<sup>6</sup>

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<sup>3</sup>By doing this the monopolist could maintain the market power, whereas competition results if the entrant obtains the patent. See Gilbert and Newbery (1982).

<sup>4</sup>I thank John Morgan for pointing out this example to me.

<sup>5</sup>For general results regarding subcontracting, please refer to Kamine et al. (1989) and Gale et al. (2000).

<sup>6</sup>For example, near the end of each year in China, a large number of migrant workers have to pay much more for train tickets than their face value in order to go back to their hometown for family reunion during the Chinese New Year. The middlemen, or *huangniu* (yellow bulls), who have lower opportunity cost of time would wait in line and make illegal profits by reselling the tickets to those who long for going home but have no time waiting in line to buy the tickets.

The above examples show that many realistic situations could be best analyzed through a model that incorporates resale possibilities into all-pay auctions. It remains open to characterize players' behaviors in all-pay auctions with resale. Intuitively existence of resale possibilities exhibits influence on bidders' bidding behaviors in the first stage. The aftermarket buyers usually have access to information revealed by the initial seller, such as submitted bids. If the submitted bids reveal private information of primary bidders, resale price will be responsive to those bids and a bidder's resale profit can depend on the bid he makes in the primary auction. Therefore, resale possibilities introduce an endogenous element to bidders' valuations upon winning the auction and creates an incentive for primary bidders to signal their private information to aftermarket buyers. This information connection between resale price and submitted bids is our primary focus.

The main objective of this paper is to investigate the effect of resale possibilities on bidders' bidding behaviors and the resulting expected revenues from both first- and second-price all-pay auctions. This paper considers a two-stage model in which an all-pay auction in the first stage is followed by resale. For the first stage, we analyze both first- and second-price all-pay auctions. For the second stage, we do not specify resale mechanism and simply assume that resale is conducted through competitive market. Therefore, winners of the first-stage auction have no bargaining power and can only affect their profits by signalling their private information through bids. This assumption could be relaxed if there is only one aftermarket buyer. For multiple buyers, this assumption is justified if interested buyers come to the market randomly and each propose a take-it-or-leave-it offer to the winner.<sup>7</sup> In certain industries such as automobile and electronics, subcontracting is based on competitive bids.<sup>8</sup>

We extend Krishna and Morgan (1997)'s general symmetric setting to incorporate resale possibilities into all-pay auctions. We characterize symmetric bidding equilibria for both

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<sup>7</sup>In equilibrium, such offer will be accepted by the winner. It makes more sense if the winner discounts future payoff more than the buyer does.

<sup>8</sup>The point is that subcontracting is used to reduce costs if we assume production cost is strictly convex.

first- and second-price all-pay auctions with resale. Based on these equilibria, we compare the two formats from the perspective of a revenue-maximizing seller. In addition, we examine the impact of information disclosure of the initial seller.

This paper is related to the literature regarding auctions with resale. Bikhchandani and Huang (1989) present a closely related model with symmetric information applicable to treasury bill auctions, where pure common values and a competitive resale market are assumed. Resale takes place because most bidders in the first stage are speculators and bid for resale. They characterize equilibrium bidding strategies for both discriminatory and uniform-price auctions. Provided existence of symmetric equilibria, they show that uniform-price auctions generate no less expected revenue than discriminatory auctions. We study all-pay auctions with resale using a similar model, but we allow bidders' valuations to be interdependent.

Using a model with independent private values, Haile (2003) studies auctions with resale under private uncertainties. Resale takes place because of the discrepancy between the estimated values at the time of bidding and the true values realized after the auction. He characterizes equilibrium bidding strategies for first-price, second-price and English auctions followed by resale which could be formalized as an optimal auction or an English auction. He argues that the option to resell creates endogenous valuations and induces signaling incentives that may revert the revenue results obtained in the literature that assumes no resale.<sup>9</sup> Assuming positively correlated signals and interdependent valuations, we show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale possibilities.

The rest of this paper is organized as following. Section 2 contains the model. Section 3 and 4 study second-price and first-price all-pay auctions respectively. Section 5 provides ranking of expected revenues of these two auction formats. Section 6 examines the effect of information disclosure. Section 7 concludes. Appendix contains all proofs.

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<sup>9</sup>Similar signalling incentive is also examined in Goeree (2003).

## 2 The Model

The game proceeds as following. In the first stage,  $N$  risk-neutral bidders compete for an indivisible good in an all-pay auction—either first-price or second-price. The bidder who submits the highest bid wins the object and pays either his own bid or the highest losing bid, depending on the exogenously chosen auction format. All losing bidders pay their own bids. Due to institutional or other reasons, there are some bidders who cannot participate in the first-stage competition.<sup>10</sup> However, they will try to obtain the object through aftermarket bargaining with the winner. Therefore, after the primary auction is over, there is possibility for resale.

In the second stage, potential buyers approach the primary winner in order to obtain the object. We assume that those losers from the first stage do not participate in the aftermarket competition. Actually in our symmetric setting of the game, if the equilibrium strategies are nondecreasing, there is no potential gain for resale taking place within the same group of bidders. On the other hand, it is likely that the valuation of certain aftermarket buyer exceeds that of primary winner, therefore there may be potential gain to be realized if new entrants bargain with the first-stage winner.

After the first-stage auction is over, we assume that the initial seller announces the winning bid and the highest losing bid.<sup>11</sup> Based on the released information, aftermarket buyers could infer the private information held by the first-stage winner. To focus on the information transmission given resale possibilities, we assume that resale price will be the expected first-stage winner's valuation of the object conditional on all publicly available information. Therefore, the seller has no bargaining power.

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<sup>10</sup>Some bidders may not be eligible to participate in certain competition for a special prize, say an monopoly privilege. Some bidders may be excluded by strategic behaviors of a revenue-maximizing seller. See an example in Section 5.

<sup>11</sup>This information release procedure is standard in most auction literature. Although interesting, the optimal information disclosure is not addressed in this paper. Calzolari and Pavan (2006) studies optimal information disclosure for a monopolist who cannot commit to prevent resale.

For the first-stage all-pay auction, we follow the framework and notation of Krishna and Morgan (1997) to make the analysis consistent with the literature. Prior to auction each bidder  $i$  receives a private signal,  $X_i$ , that affects value of the object  $V_i$  defined as:

$$V_i = V(S, X_i, \{X_j\}_{j \neq i}) \quad (1)$$

where  $S = (S_1, S_2, \dots, S_m)$  are any other random variables that influence the valuation but are not observed by any bidder. We assume that  $V$  is non-negative, continuous, increasing in all its variables. For each  $i$ ,  $E[V_i] < \infty$ . Moreover, all bidders' valuations depend on  $S$  in the same manner, and each bidder's valuation is a symmetric function of other bidders' signals.

Let  $X_0$  be private information held by the initial seller who may or may not reveal it. Let  $f(S, X_0, X_1, X_2, \dots, X_n)$  be the joint density of random variables  $S, X_0, X_1, X_2, \dots, X_n$ , where  $f$  is symmetric in bidders' signals. We assume that  $f$  satisfies the affiliation inequality:

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z') \quad (2)$$

where  $z \vee z'$  denotes the component-wise maximum of  $z$  and  $z'$  and  $z \wedge z'$  denotes the component-wise minimum of  $z$  and  $z'$ . Roughly, this means that a high value of one of the variables,  $S_j$  or  $X_i$ , makes it more likely that the other variables also take on high values. Let  $[0, \bar{s}]^m \times [0, \bar{x}_0] \times [0, \bar{x}]^n$  be the support of  $f$ , where  $[0, \bar{x}]^n$  denotes the  $n$ -fold product of  $[0, \bar{x}]$ .

Let  $f_{Y_1}(\cdot|x)$  denote the conditional density of  $Y_1$ , where  $Y_1 = \max\{X_j\}_{j \neq 1}$ , given  $X_1 = x$ . Standard results from Milgrom and Weber (1982) show that  $X_1$  and  $Y_1$  are also affiliated. Throughout the paper, we make use of the following facts:  $F_{Y_1}(y|x)$  and  $\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$  are non-increasing in  $x$ .<sup>12</sup> Moreover, if  $H$  is any nondecreasing function, affiliation implies that  $h(a_1, b_1; \dots, a_n, b_n) = E[H(X_1, \dots, X_n) | a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n]$  is nondecreasing in all of its arguments. For simplicity, we also assume that if  $H$  is continuously differentiable, then

<sup>12</sup>In the appendix, we provide the detailed proof for these facts.

$E[H(X_1, \dots, X_n | a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n)]$  is also continuously differentiable in all its arguments.<sup>13</sup>

A *pure strategy* for bidder  $i$  is a measurable function,  $\beta_i : [0, \bar{x}] \rightarrow \mathbb{R}$ . Such a pure strategy is *monotone* if  $x' \geq x$  implies  $\beta_i(x') \geq \beta_i(x)$ .

An N-tuple of pure strategies,  $(\beta_1, \dots, \beta_n)$  is an *equilibrium* if for every bidder  $i$  and every pure strategy  $\beta'_i$ ,

$$EU_i(\beta(x), x) \geq EU_i(\beta'_i(x), \beta_{-i}(x_{-i}), x) \quad (3)$$

where the left-hand side, is bidder  $i$ 's expected utility given the joint strategy  $\beta$ , and the right-hand side is his expected utility when he employs  $\beta'_i$  and the others employ  $\beta_{-i}$ .

The equilibrium is *symmetric* if  $\beta_1 = \dots = \beta_N = \beta$ . Since bidders are ex ante identical, we are considering symmetric equilibrium bidding strategies.

### 3 Second-Price All-Pay Auctions with Resale

We first characterize the symmetric equilibrium for second-price all-pay auction with resale. Without loss of generality, we analyze the game from bidder 1's point of view. When bidder 1 submits his bid, he only observes his own private signal  $X_1$ .

According to our assumption, buyers on the secondary market observe only publicly announced information and resale price is the expectation of the primary winner's valuation conditional on all public information. It is useful to begin with a heuristic derivation of the first-order condition for  $\beta_s$  to be a symmetric Nash equilibrium in strictly increasing and differentiable strategies.<sup>14</sup>

Suppose bidders  $j \neq 1$  follow the symmetric equilibrium strategy  $\beta_s$ . Suppose bidder 1 receives a private signal  $X_1 = x$  and bids  $b$ . If bidder 1 wins with a bid  $b$  and the secondary

<sup>13</sup>For more details about affiliation, please refer to Milgrom and Weber (1982).

<sup>14</sup>We will show later that the equilibrium strategy is indeed strictly increasing and differentiable.

market buyers believe that he is following  $\beta_s$ , the resale price will be

$$P(\beta_s^{-1}(b), Y_1) = E[V_1 | X_1 = \beta_s^{-1}(b), Y_1] \quad (4)$$

where  $\beta_s^{-1}$  denotes the inverse of  $\beta_s$  and  $Y_1$  is the first-order statistic of  $(X_2, \dots, X_N)$ . When bidder 1 wins the object and buyers on the secondary market believe that his private signal is equal to  $x'$ , the expected resale price conditional on  $X_1$  and  $Y_1$  is:

$$v(x', x, y) \equiv E[P(x', Y_1) | X_1 = x, Y_1 = y] \quad (5)$$

By affiliation, both  $P$  and  $v$  are non-decreasing in all their arguments. With this notation, the expected payoff for bidder 1 is:

$$\Pi(b, x) = \int_0^{\beta_s^{-1}(b)} (v(\beta_s^{-1}(b), x, y) - \beta_s(y)) f_{Y_1}(y|x) dy - [1 - F_{Y_1}(\beta_s^{-1}(b)|x)] b \quad (6)$$

Maximizing (6) with respect to  $b$  yields the first-order condition

$$\begin{aligned} 0 &= \frac{1}{\beta_s'(\beta_s^{-1}(b))} v(\beta_s^{-1}(b), x, \beta_s^{-1}(b)) f_{Y_1}(\beta_s^{-1}(b)|x) \\ &\quad + \frac{1}{\beta_s'(\beta_s^{-1}(b))} \int_0^{\beta_s^{-1}(b)} v_1(\beta_s^{-1}(b), x, y) f_{Y_1}(y|x) dy \\ &\quad - [1 - F_{Y_1}(\beta_s^{-1}(b)|x)] \end{aligned}$$

where  $v_1$  is the partial derivative with respect to  $\beta_s^{-1}(b)$ .

At a symmetric equilibrium, it is optimal that  $\beta_s(x) = b$ , then we have

$$\beta_s'(x) = v(x, x, x) \frac{f_{Y_1}(x|x)}{1 - F_{Y_1}(x|x)} + \int_0^x v_1(x, x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(x|x)} dy \quad (7)$$

The solution with the boundary condition  $\beta_s(0) = 0$  is:

$$\beta_s(x) = \int_0^x v(t, t, t) \frac{f_{Y_1}(t|t)}{1 - F_{Y_1}(t|t)} dt + \int_0^x k(u) du \quad (8)$$

where  $k(u) = \int_0^u v_1(u, u, y) \frac{f_{Y_1}(y|u)}{1 - F_{Y_1}(u|u)} dy$ .

This is only necessary condition for a symmetric equilibrium. For sufficiency, we need the following assumption.

**Assumption 1.** Let  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by  $\psi(x', x, y) = v(x', x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)}$ . We assume that for all  $y$ , (i)  $\psi_2 > 0$ , and (ii)  $\psi_{12} > 0$ .

Given the above assumption, we have the following result.

**Theorem 1.** Suppose Assumption 1 hold, then a symmetric equilibrium of second-price all-pay auction with resale is given by  $\beta_s$  defined as

$$\beta_s(x) = \int_0^x v(t, t, t) \frac{f_{Y_1}(t|t)}{1 - F_{Y_1}(t|t)} dt + \int_0^x k(u) du$$

where  $k(u) = \int_0^u v_1(u, u, y) \frac{f_{Y_1}(y|u)}{1 - F_{Y_1}(u|u)} dy$ .

**Remark 1.** Assumption 1 is restrictive and critical to the sufficiency result. Note that affiliation implies that  $\frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)}$  is non-increasing in  $x$ , and  $v(x', x, y)$  is non-decreasing in  $x$  for every  $y$ . Therefore, part (i) ensures that the affiliation between  $X_1$  and  $Y_1$  is not so strong that it overwhelms the increase in the expected valuation of the object,  $v(x', x, y)$ , resulting from a higher signal  $x$ . Part (ii) is needed to ensure that the responsiveness of resale price and primary bidders' signalling incentive are increasing in signal.

**Example 1.** Suppose  $N = 2$ . Let  $f(x, y) = \frac{4}{9}(2 + xy)$  on  $[0, 1]^2$ , and  $v(x', x, y) = x'x + \frac{1}{2}y$ . Simple manipulation yields  $f_y(y|x) = \frac{4+2xy}{4+x}$ , and  $F_Y(y|x) = \frac{4y+xy^2}{4+x}$ . Then we have

$$\psi(x', x, y) = v(x', x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)} = \frac{(2x'x + y)(2 + xy)}{4 + x - 4y - xy^2}$$

*It can be verified that all conditions in Assumption 1 is satisfied.*

Using a variation of part (i) of Assumption 1, Krishna and Morgan (1997) derive sufficient condition for a symmetric equilibrium of second-price all-pay auction (the war of attrition) without resale. In their context,  $v(x, y) = E[V_1 | X_1 = x, Y_1 = y]$ , and the symmetric equilibrium is given by:

$$\alpha_s(x) = \int_0^x v(t, t) \frac{f_{Y_1}(t|t)}{1 - F_{Y_1}(t|t)} dt \quad (9)$$

To ensure it is indeed a symmetric equilibrium, they assume that  $\phi(\cdot, y)$  is increasing for all  $y$ , where  $\phi(x, y) \equiv v(x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)}$ .

Examining both equilibria, we could observe that  $\beta_s(x)$  reduces to  $\alpha_s(x)$  if there is no resale possibilities and the primary bidders have no incentive to signal. Resale possibilities introduce an endogenous element to primary bidders' valuation since bidders' resale profit depend on the bids they make in the primary auction. Hence primary bidders have incentives to signal their private information to aftermarket buyers. They signal in order to convince aftermarket buyers that the object is of high value since resale price is responsive to the announced bids. This responsiveness is measured by  $v_1$ ;  $v_1$  is nonnegative and increasing in  $x$  due to affiliation and Assumption 1.

From the analysis above, we can conclude that the information disclosing policy is crucial to our characterization. In second-price all-pay auctions with resale, if only the highest losing bid (price paid by the winner) is revealed, primary bidders' signalling incentive will be reduced since the winning bid conveys private information for the first-stage winner. On the other hand, if the initial seller releases more information, the expected resale price will not decrease, may increase based on more information. This increases the expected valuation of bidders upon winning, hence they will bid more aggressively than otherwise.

## 4 First-Price All-pay Auctions with Resale

The analysis is parallel to previous section. Again we begin with a heuristic derivation of equilibrium. Suppose bidders  $j \neq 1$  follow the symmetric equilibrium strategy  $\beta_f$ . Suppose bidder 1 receives a private signal  $X_1 = x$  and bids  $b$ . If bidder 1 wins with a bid  $b$  and aftermarket buyers believe that he is following  $\beta_f$ , slightly abusing notation yields

$$P(\beta_f^{-1}(b), Y_1) = E[V_1 | X_1 = \beta_f^{-1}(b), Y_1]$$

where  $\beta_f^{-1}$  denotes the inverse of  $\beta_f$ . When bidder 1 wins the object and buyers on the secondary market believe that his private signal is equal to  $x'$ , the expected resale price conditional on  $X_1$  and  $Y_1$  is:

$$v(x', x, y) \equiv E[P(x', Y_1) | X_1 = x, Y_1 = y]$$

By affiliation, both  $P$  and  $v$  are non-decreasing in all their arguments. With this notation, the expected payoff for bidder 1 is

$$\Pi(b, x) = \int_0^{\beta_f^{-1}(b)} v(\beta_f^{-1}(b), x, y) f_{Y_1}(y|x) dy - b \quad (10)$$

Maximizing (10) with respect to  $b$  yields the first-order condition:

$$\begin{aligned} 0 &= v(\beta_f^{-1}(b), x, \beta_f^{-1}(b)) f_{Y_1}(\beta_f^{-1}(b)|x) \frac{1}{\beta'_f(\beta_f^{-1}(b))} \\ &+ \frac{1}{\beta'_f(\beta_f^{-1}(b))} \int_0^{\beta_f^{-1}(b)} v_1(\beta_f^{-1}(b), x, y) f_{Y_1}(y|x) dy - 1 \end{aligned}$$

where  $\beta'_f$  is the first derivative of  $\beta_f$ , and  $v_1(\beta_f^{-1}(b), x, y)$  is the partial derivative of  $v$  with respect to its first argument.

At a symmetric equilibrium,  $\beta_f(x) = b$  and thus

$$\beta'_f(x) = v(x, x, x) f_{Y_1}(x|x) + \int_0^x v_1(x, x, y) f_{Y_1}(y|x) dy \quad (11)$$

The solution to equation (11) with the boundary condition  $\beta_f(0) = 0$  is:

$$\beta_f(x) = \int_0^x v(t, t, t) f_{Y_1}(t|t) dt + \int_0^x h(u) du \quad (12)$$

where  $h(u) = \int_0^u v_1(u, u, y) f_{Y_1}(y|u) dy$ .

The derivation is heuristic since (11) is only a necessary condition. For the sufficiency, we need additional restriction like Assumption 1. Let  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$\Phi(x', x, y) = v(x', x, y) f_{Y_1}(y|x)$$

One implication of Assumption 1 leads to the following lemma. The argument makes use of the fact that  $F_{Y_1}(y|x)$  is non-increasing in  $x$ , and proof is contained in appendix.

**Lemma 1.** *Suppose Assumption 1 hold. Then for all  $y$ , we have (i)  $\Phi_2 > 0$ , and (ii)  $\Phi_{12} > 0$ .*

With Lemma 1, we can show that the equilibrium we characterize is indeed a symmetric equilibrium.

**Theorem 2.** *Suppose Assumption 1 hold, then a symmetric equilibrium in first-price all-pay auctions with resale is given by  $\beta_f$  defined as*

$$\beta_f(x) = \int_0^x v(t, t, t) f_{Y_1}(t|t) dt + \int_0^x h(u) du$$

where  $h(u) = \int_0^u v_1(u, u, y) f_{Y_1}(y|u) dy$ .

For first-price all-pay auctions without resale, Krishna and Morgan (1997) characterize a

symmetric equilibrium:

$$\alpha_f(x) = \int_0^x v(t, t) f_{Y_1}(t|t) dt \quad (13)$$

To ensure it is indeed a symmetric equilibrium, they assume that  $\varphi(\cdot, y)$  is increasing for all  $y$ , where  $\varphi(x, y) \equiv v(x, y) f_{Y_1}(y|x)$ .<sup>15</sup>

Examining both equilibria, we find that  $\beta_f(x)$  reduces to  $\alpha_f(x)$  if there is no resale possibility and primary bidders have no incentive to signal. As we argue in previous section, resale possibilities introduce an endogenous element to primary bidders' valuation since bidders' resale profit depend on the bids they make in the primary auction. Hence primary bidders have incentives to signal their private information to aftermarket buyers. The implication of Assumption 1, say Lemma 1, ensures that the responsiveness of resale price to announced bids (measured by  $v_1$ ) increases with a bidder's private signal. This further guarantees that each bidder's incentive to signal increases with his private signal. Without resale, only  $\Phi_2 > 0$  is needed to characterize the symmetric equilibrium.

Because of responsiveness of resale price to announced bids, the information disclosing policy affects primary bidders' bidding behaviors. Affiliation implies that resale price is non-decreasing in all bidders' private signals. Therefore, the first-stage bidders may bid more aggressively if the initial seller announces more bids. If the initial seller announces less information, bidders' incentives to signal their private information will be reduced.

## 5 Revenue Comparison

In this section, we investigate the performance of first- and second-price all-pay auctions with resale in terms of expected revenue accruing to the initial seller. Given the symmetric equilibria we characterize, which auction format is better from the perspective of a revenue-maximizing seller? Without resale, Krishna and Morgan (1997) derive a revenue ranking

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<sup>15</sup>Krishna and Morgan (1997) show that for all  $y$ , that  $\phi(\cdot, y)$  is increasing implies  $\varphi(\cdot, y)$  is increasing.

between these two auction formats: second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions.<sup>16</sup> The following subsection examines whether this ranking remains true with resale possibilities.

## 5.1 Two All-Pay Auction Formats with Resale

First, let us compare the expected revenue generated by both auction formats at symmetric equilibria.

**Theorem 3.** *Suppose Assumption 1 hold. With resale possibilities, the expected revenue from second-price all-pay auction is greater than or equal to that from first-price all-pay auction at the symmetric equilibria.*

The proof is contained in the appendix. Here we provide an intuitive explanation using *linkage principle*. Milgrom and Weber (1982) originally introduce linkage principle to auction literature in order to derive the revenue ranking among first-price, second-price, and English auction when signals are affiliated and valuations are interdependent. One of the implications is that the expected revenue from second-price auctions is no less than that from first-price auctions. Krishna and Morgan (1997) further apply linkage principle to all-pay auctions and show that this ranking remains true if we require all bidders pay their bids. Theorem 3 further implies that this revenue ranking maintains when there are resale possibilities. The common thread running through is *linkage principle*.

Consider the auctions as revelation games, then the selling price (the revenue of seller) could depend only on the bids or bidders' reports, and on the seller's information. Then if the winner's payment depend on the second-highest bidder's signal, which is affiliated with the winner's private signal, the expected payment would be statistically linked to that information. As a result of affiliation, this linkage reduces the information rent the seller must

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<sup>16</sup>They further show that second-price all-pay auctions generate no less expected revenue than second-price auctions, and first-price all-pay auctions generate no less expected revenue than first-price auctions.

leave to bidders to induce truthful revelation of private information. For a fixed bid, a higher private signal of the winner means that the second highest signal is more likely to take on high values, so the expected payment of the winner is also higher. Hence, the linkage makes the expected price paid in equilibrium by the winner increase more steeply as a function of his signal than otherwise. By our boundary condition, a bidder with the lowest type pays nothing, then a steeper expected payment function yields higher expected prices. This is the intuition behind the revenue ranking result. Clearly this linkage still works when resale is allowed.

To see how the linkage principle work, suppose bidder 1 learns his private signal as  $x$ , but bids as if it were  $z$ . Let  $e^k(z, x)$  denote the expected payment made by bidder 1 in the  $k$ -price all-pay auction with resale, where  $k = \{f, s\}$ .

Krishna and Morgan (1997) derive a variation of linkage principle that is more useful in our model of all-pay auctions with resale.

**Proposition 1.** *Suppose  $L$  and  $M$  are two auction mechanisms with symmetric increasing equilibria such that the expected payment of a bidder with the lowest signal is 0. If for all  $x$ ,  $e_2^M(x, x) \geq e_2^L(x, x)$  then for all  $x$ ,  $e^M(x, x) \geq e^L(x, x)$ .*

To see how this principle works, let  $R(z, x)$  denote the expected value received by bidder 1. Then

$$R(z, x) = \int_0^z v(z, x, y) f_{Y_1}(y|x) dy \quad (14)$$

Indeed, this expression is the same for both all-pay auction forms at the symmetric equilibria. Then the expected payoff for bidder 1 is:

$$\Pi^k(z, x) = R(z, x) - e^k(z, x) \quad (15)$$

In equilibrium, it is optimal to choose  $z = x$ , and the first-order condition yields  $e_1^f(x, x) = e_1^s(x, x)$ , where  $e_1^k(x, x)$  is the derivative of  $e^k(z, x)$  with respect to  $z$  evaluated at  $z = x$ .

Note that we have

$$e^k(z, x) = \begin{cases} e^f(z) = \beta_f(z), & \text{if } k = f \\ \int_0^z \beta_s(y) f_{Y_1}(y|x) dy + [1 - F_{Y_1}(z|x)] \beta_s(z), & \text{if } k = s \end{cases}$$

Taking the derivative with respect to  $x$ , it is trivial to show that  $e_2^f(z, x) = 0$  and

$$e_2^s(z, x) = -\frac{\partial}{\partial x} \int_0^z \beta_s'(y) F_{Y_1}(y|x) dy \geq 0$$

since  $F_{Y_1}(y|x)$  is non-increasing in  $x$  due to affiliation. Applying Proposition 1, we have  $e^s(x, x) \geq e^f(x, x)$  since  $e^s(0, 0) = e^f(0, 0) = 0$ .

Therefore, the initial seller could benefit from exogenously choosing second-price instead of first-price all-pay auctions if resale is allowed.

## 5.2 Does Resale Benefit Seller?

There are still some questions to be explored. First, if the seller can commit to allow resale or not, should he allow resale? Under what conditions does the existence of an active secondary market benefit the initial seller?<sup>17</sup> Generally, if the first-stage winner has access to some potential buyers to bargain, primary participants not only compete for the object, but also compete for the right to resell. By signalling their private information, they bid more aggressively than without resale possibilities.

Second, does a revenue-maximizing seller have incentive to exclude some bidders from the first stage competition, forcing them to the secondary market? This question remains open with affiliated signals and interdependent values. In the setting of independent private values, we can provide a partial answer using the following example.

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<sup>17</sup>Calzolari and Pavan (2006) show that a monopolist benefits from the existence of resale market when he cannot contract with all potential buyers and he can prohibit the winner from reselling to the losers. However, the monopolist will get hurt if resale cannot be banned and takes place among the same group of bidders.

**Example 2.** Suppose there are  $N$  risk-neutral bidders competing over an indivisible prize in a first-price all-pay auction. Each bidder has independent private valuation from identical uniform distribution  $F$  on  $[0, 1]$ .

For simplicity, there are no new entrant coming to the auction, but the initial seller may exclude bidders from competition, forcing them into the resale market. Without exclusion, the revenue equivalence theorem yields that the expected revenue ( $R_N$ ) is the expected value of the second highest valuation among  $N$  bidders. Hence, we have

$$R_N = E[Y_1] = \frac{N-1}{N+1}$$

Using revenue equivalent theorem, we also have the symmetric equilibrium strategy

$$\beta_N(x) = \int_0^x yg(y)dy$$

where  $g(\cdot)$  is the density of the distribution of  $Y_1^{N-1}$ , the highest order statistics among  $N-1$  variables. Now suppose one of the bidders is excluded randomly, then this exclusion creates an opportunity for the winner to resell in the secondary market. In this example, the valuations are private, so there is no signalling incentive. However, resale possibility still introduces an endogenous element to bidders' valuation since the winning bidder could extract more surplus by reselling the object to the excluded bidder. Suppose the winner of primary competition has full bargaining power and extract the whole surplus from resale. Then, upon winning, the winner's valuation becomes:

$$\begin{aligned} v(x) &= xF(x) + \int_x^1 [x + (s-x)]f(s)ds \\ &= \frac{1}{2}(1+x^2) \end{aligned}$$

It is trivial to verify that the equilibrium strategy becomes

$$\beta_{N-1}(x) = \int_0^x \frac{1+z^2}{2} h(z) dz$$

where  $z$  is the realization of  $Z_1 \equiv Y_1^{N-2}$ , the highest order statistics among  $N-2$  variables, and  $h(\cdot)$  denotes the density of the distribution of  $Z_1$ . Therefore,

$$\begin{aligned} R_{N-1} &= (N-1) \int_0^1 \left[ \int_0^x \left( \frac{1+z^2}{2} \right) h(z) dz \right] dx \\ &= \frac{N^2 - N + 1}{N(N+1)} \end{aligned}$$

Obviously  $R_{N-1} - R_N = \frac{1}{N(N+1)} > 0$ . Therefore, the initial seller could benefit by excluding one bidder from the first-stage competition and introducing resale possibilities.

Baye et al. (1993) present an interesting *exclusion principle*: a revenue maximizing politician may find it in his best interest to exclude lobbyists with valuations above a threshold from participating in the all-pay auction. Since values are public information, the exclusion makes the competition more even and bidders submit higher bids, which in turn increases the initial seller's expected revenue. Bose and Deltas (1999) study English auction with two distinct types of potential bidders: consumers who bid for their own consumption and speculators who bid for resale. They show that, if the speculators have access to a larger market of consumers than the seller, then the seller prefer to prevent the consumers from participating in the auction.

Haile (1996) derives similar exclusion result for a second-price auction followed by resale. In particular, he shows that as long as the first-stage winner extracts no less than two thirds of resale surplus, the initial seller could benefit by excluding one bidder from the first-stage competition. In our example, the same result holds, that is, as long as the winner extracts at least two thirds of resale surplus, the seller finds it profitable to exclude a bidder randomly.

By the same argument, the seller can continue excluding one more bidder, provided that the winner gets more bargaining power. Indeed, it can be shown that the proportion of resale surplus accruing to the winner is strictly increasing in the number of bidders excluded by the seller. Intuitively, the more bargaining power the winner has, the more aggressively he bids in the first stage. Therefore, the expected revenue increases as the seller excludes more bidders and the seller excludes more bidders. If the winner's bargaining power is bounded, there will be an optimal exclusion policy for the seller.

## 6 Information Disclosure by the Initial Seller

Very often the initial seller has private information that may affect bidders' valuation or private information. Suppose the initial seller has private signal  $X_0$  that is affiliated with all bidders' signals. Now consider how equilibria in all-pay auctions are affected when the initial seller publicly reveals  $X_0$ . Conditional on  $X_0 = x_0$ , we could derive the symmetric equilibria for both second-price and first-price all-pay auctions with resale.

Before stating the results, we need more notations. Let  $\tilde{\beta}_k(\cdot, x_0)$  be a symmetric equilibrium bidding strategy conditional on  $X_0 = x_0$ ,  $k = \{f, s\}$ . Define

$$P(\tilde{\beta}_k^{-1}(b, x_0), Y_1, X_0) = E[V_1 | X_1 = \tilde{\beta}_k^{-1}(b, x_0), Y_1, X_0 = x_0] \quad (16)$$

as the resale price if bidder 1 wins the auction with a bid  $b$ , and aftermarket buyers believe that he is following  $\tilde{\beta}_k(\cdot, x_0)$ . Similarly, define

$$\tilde{v}(x', x, y, x_0) = E[P(x', Y_1, X_0) | X_1 = x, Y_1 = y, X_0 = x_0] \quad (17)$$

as the expected resale price conditional on  $X_1, Y_1$  and  $X_0$ . To derive the symmetric equilibria given  $X_0 = x_0$ , a modification of Assumption 1 is needed.

**Assumption 2.** Let  $\tilde{\Psi} : \mathbb{R}^4 \rightarrow \mathbb{R}$  be defined by  $\tilde{\Psi}(x', x, y, x_0) = \tilde{v}(x', x, y, x_0) \frac{f_{Y_1}(y|x, x_0)}{1 - F_{Y_1}(y|x, x_0)}$ . We assume that for all  $y$ , (i)  $\tilde{\Psi}_2 > 0, \tilde{\Psi}_4 > 0$ , and (ii)  $\tilde{\Psi}_{12} > 0, \tilde{\Psi}_{14} > 0$ .

By the same argument as Lemma 1, we have

**Lemma 2.** Suppose Assumption 2 hold. Let  $\tilde{\Phi} : \mathbb{R}^4 \rightarrow \mathbb{R}$  be defined by  $\tilde{\Phi}(x', x, y, x_0) = \tilde{v}(x', x, y, x_0) f_{Y_1}(y|x, x_0)$ , then for all  $y$ , (i)  $\tilde{\Phi}_2 > 0, \tilde{\Phi}_4 > 0$ , and (ii)  $\tilde{\Phi}_{12} > 0, \tilde{\Phi}_{14} > 0$ .

As we characterize the symmetric equilibria without information about the seller's private signal, we could derive the symmetric equilibria conditional on seller's private information for both auction formats.

**Proposition 2.** Suppose Assumption 2 hold. Conditional on the seller's private signal  $X_0$ , a symmetric equilibrium in second-price all-pay auctions with resale is given by  $\tilde{\beta}_s$  defined as

$$\tilde{\beta}_s(x, x_0) = \int_0^x \tilde{v}(t, t, t, x_0) \frac{f_{Y_1}(t|t, x_0)}{1 - F_{Y_1}(t|t, x_0)} dt + \int_0^x \tilde{k}(u) du \quad (18)$$

where  $\tilde{k}(u) = \int_0^u \tilde{v}_1(u, u, y, x_0) \frac{f_{Y_1}(y|u, x_0)}{1 - F_{Y_1}(u|u, x_0)} dy$ .

**Proposition 3.** Suppose Assumption 2 hold. Conditional on the seller's private signal  $X_0$ , a symmetric equilibrium in first-price all-pay auctions with resale is given by  $\tilde{\beta}_f$  defined as

$$\tilde{\beta}_f(x, x_0) = \int_0^x \tilde{v}(t, t, t, x_0) f_{Y_1}(t|t, x_0) dt + \int_0^x \tilde{h}(u) du \quad (19)$$

where  $\tilde{h}(u) = \int_0^u \tilde{v}_1(u, u, y, x_0) f_{Y_1}(y|u, x_0) dy$ .

**Remark 2.** Note that the equilibrium bidding function  $\tilde{\beta}$  now maps two variables into a bid. For any fixed value of  $X_0$ , the equilibrium bidding strategy is a function of bidder's private signal only and is similar to  $\beta$ . Affiliation between  $X_0$  and  $(X_1, \dots, X_N)$ , Assumption 2 and Lemma 2 guarantee that the equilibrium bidding function  $\tilde{\beta}$  is increasing as  $X_0$  increases. Conditional on  $X_0$ , primary bidders signal their private information to aftermarket buyers

*through their bids. The responsiveness of resale price to announced bids and information increases as the realization of a bidder's private signal.*

An immediate implication of affiliation and Assumption 2 is that the initial seller could benefit from publicly releasing his private signal.

**Proposition 4.** *Suppose that Assumption 2 hold. A policy of publicly revealing the initial seller's private information cannot lower, and may raise the expected revenue for the seller in all-pay auctions with resale.*

The intuition underlying this result can be best understood through linkage principle. Publicly releasing his private signal, the initial seller establishes a link between the bids submitted and that signal. This additional link reduces the information rent enjoyed by the bidders possessing private information. Hence, the revenue-enhancing result follows as a consequence of linkage principle. Therefore, releasing the seller's private signal has similar effect as releasing more bids.

Obviously Proposition 4 relies crucially on Assumption 2. If Assumption 2 fails to hold,  $\tilde{\beta}(x, x_0)$  may not be an increasing function of  $x_0$  because the marginal effect of the bid on the resale price may be reduced and revealing  $X_0$  may reduce the bidders' incentive to signal, and then lower the expected revenue for the seller even though  $X_0, X_1, \dots, X_N$  are affiliated. Extremely, if  $X_0$  contains all the relevant information in  $X_1, X_2, \dots, X_N$ , there will be no signaling incentive for the bidders, and the expected price in all-pay auctions will be lower than otherwise. The various information structure and corresponding optimal information disclosure policy is technically complex and remains open. Intuitively the optimal information disclosure policy depends on the specific resale mechanism and the distribution of bargaining power between the winner and aftermarket buyers.<sup>18</sup>

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<sup>18</sup>Calzolari and Pavan (2006) studies optimal information disclosure for a monopolist who cannot commit to prevent resale.

## 7 Conclusions

This paper studies all-pay auctions with resale. Costly competitions over a limited number of prizes are often followed by aftermarket interaction, as winners of patent races sell or license patents to other producers. We find that introducing resale possibilities changes bidders' behaviors in all-pay auctions. The information connection between the resale prices and the bids submitted by the first-stage bidders creates signalling incentive for primary bidders. We provide sufficient conditions under which symmetric equilibria exist and characterize equilibria strategies. Provided the existence of symmetric equilibria, we show that second-price all-pay auctions generate no less expected revenue than first-price all-pay auctions with resale. Furthermore, if the bidders' signals are affiliated to the initial seller's private signal, the seller could enhance his expected revenue by publicly disclosing that information.

Several extensions of this model are as following. First, we do not explicitly formalize the resale mechanism. We simply assume that the resale price equals to the expected valuation of the winner in the first stage. In practice, the resale mechanism could be another auction, or a multilateral bargaining. Intuitively different resale mechanisms and the distribution of bargaining power will affect the split of resale surplus between resale seller and buyers, and in turn affect the bidding strategies adopted by primary bidders.<sup>19</sup>

Second, we assume that the initial seller announce the winning bid and the highest losing bid after the primary auction. Obviously different information disclosing policies have different impacts on the significance of information linkage between resale price and submitted bids. The signalling behavior relies on the announcement of winning bids. Therefore, it remains a challenging question to investigate the optimal information disclosing policy from the seller's perspective. From the standpoint of mechanism design, the optimal auction with resale may also depend on the information disclosing policy. The characterization of optimal

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<sup>19</sup>Using an asymmetric two-bidder independent private value model, Hafalir and Krishna (2006) characterize the equilibrium bidding strategies when resale takes place via monopoly pricing. They also show that the results could easily extend to other resale mechanisms such as monopsony pricing and a probabilistic  $k$ -double auction.

selling mechanism with resale seems to be another challenging exercise.<sup>20</sup>

Third, we assume that the first-stage losers do not participate in aftermarket competition. Resale takes place when new entrants come to the market. If there are a fixed number of competitors, it will be interesting to examine the optimal excluding policy from the seller's perspective. We provide a simple example to illustrate that the seller may find in his best interest to exclude one bidder randomly. A more general analysis is worth exploring.

A special characteristic of all-pay auctions lies in the deterministic winning probabilities. Many interesting instances, however, have stochastic allocation of the objects. If the winning probability is stochastic, the more a bidder bids, the higher the probability he wins. But he never guarantees winning. Then the allocation will be ex post inefficient with positive probability. It will be worth investigating whether resale enhances allocative efficiency and compare expected revenue resulting from the deterministic model with that resulting from the stochastic model.<sup>21</sup>

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<sup>20</sup>Ausubel and Cramton (1999) consider an optimal multi-unit auction with efficient secondary market. Zheng (2002) extends Myerson (1981)'s optimal auction design to the case in which resale cannot be prevented. Lebrun (2005) shows that the second-price auction with resale implements Myerson's optimal auction.

<sup>21</sup>Using a simple two-bidder-two-value stochastic model, Sui (2006) shows that resale enhances allocative efficiency and increases expected revenue for the seller as long as the winner has more bargaining power than the loser.

## A Some Facts from Affiliation

In this section, we prove some useful facts due to affiliation.

*Fact 1.*  $\frac{F_{Y_1}(y|x)}{f_{Y_1}(y|x)}$  is non-increasing in  $x$ .

*Proof.* Let  $x < x'$  and  $y < y'$ . By affiliation inequality, we have

$$f(x, y)f(x', y') \geq f(x, y')f(x', y)$$

Hence

$$\frac{f(x, y)}{f(x, y')} \geq \frac{f(x', y)}{f(x', y')}$$

Then we have

$$\frac{f_{Y_1}(y|x)}{f_{Y_1}(y'|x)} \geq \frac{f_{Y_1}(y|x')}{f_{Y_1}(y'|x')}$$

Integrating with respect to  $y$  over  $[0, y']$  yields

$$\frac{F_{Y_1}(y'|x)}{f_{Y_1}(y'|x)} \geq \frac{F_{Y_1}(y'|x')}{f_{Y_1}(y'|x')}$$

therefore the result follows. □

*Fact 2.*  $\frac{f_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$  is non-increasing in  $x$ .

*Proof.* By *Fact 1*,  $\frac{f_{Y_1}(y|x)}{F_{Y_1}(y|x)}$  is non-decreasing in  $x$ . Hence,  $-\frac{f_{Y_1}(y|x)}{F_{Y_1}(y|x)}$  is non-increasing in  $x$ .

Therefore, the result follows. □

*Fact 3.*  $F_{Y_1}(y|x)$  is non-increasing in  $x$ .

*Proof.* *Fact 1* and *Fact 2* imply that  $\frac{F_{Y_1}(y|x)}{1-F_{Y_1}(y|x)}$  is non-increasing in  $x$ . Hence  $\frac{1-F_{Y_1}(y|x)}{F_{Y_1}(y|x)}$  is non-decreasing in  $x$ . Therefore,  $\frac{1}{F_{Y_1}(y|x)}$  is non-decreasing in  $x$ . The result follows. □

## B Proof of Results

*Proof of Theorem 1*

*Proof.* The necessity is established in Section 3. For sufficiency, let  $z \leq x$ , and  $\beta_s(z) = b$ .

From the first order condition, we have

$$\begin{aligned} \frac{\partial \Pi(\beta_s(z), x)}{\partial b} &= v(z, x, z) \frac{f_{Y_1}(z|x)}{1 - F_{Y_1}(z|x)} \frac{1}{\beta'_s(z)} + \frac{1}{\beta'_s(z)} \int_0^z v_1(z, x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(z|x)} dy - 1 \\ &\geq v(z, z, z) \frac{f_{Y_1}(z|z)}{1 - F_{Y_1}(z|z)} \frac{1}{\beta'_s(z)} + \frac{1}{\beta'_s(z)} \int_0^z v_1(z, z, y) \frac{f_{Y_1}(y|z)}{1 - F_{Y_1}(z|z)} dy - 1 \\ &= \frac{\partial \Pi(\beta_s(z), z)}{\partial b} = 0 \end{aligned}$$

The first inequality follows from Assumption 1, and the last two equalities follow from the first-order condition. That means, when  $X_1 = x$  and bidder 1 bids  $b = \beta_s(z) \leq \beta_s(x)$ , his expected profit could be raised by bidding higher. By similar argument, when  $z \geq x$ , we can show  $\frac{\partial \Pi(\beta_s(z), x)}{\partial b} \leq 0$ . Consequently,  $\Pi(b, x)$  is maximized at  $\beta_s(x) = b$ . Since  $\Pi(0, x) = 0$  for all  $x$ , we have  $\Pi(\beta_s(x), x) \geq 0$  for all  $x > 0$  by affiliation. Thus, we have shown that  $\beta_s(x)$  is the best response strategy for bidder 1 when he observes  $X_1 = x$  and all other bidders  $j \neq i$  follow  $\beta_s$ , and when resale market participants believe that all bidders follow  $\beta_s$ .

From the above argument, the equilibrium payoff to a bidder who receives a signal of  $x$  is  $\Pi(\beta_s(x), x) \geq 0$ , and thus it is individually rational for each bidder to participate in the auction.

It remains to show that the equilibrium bidding strategy is strictly increasing and differentiable. Since  $v_1$  is positive by affiliation,  $\beta'_s(x)$  is strictly positive. Clearly, the equilibrium bidding strategy is differentiable. Therefore, the bidding strategy we characterize is indeed a symmetric equilibrium provided Assumption 1 holds.  $\square$

*Proof of Lemma 1*

*Proof.* Let  $x < z$ . Since  $\Psi_2 > 0$ , we have that

$$v(x', x, y) \frac{f_{Y_1}(y|x)}{1 - F_{Y_1}(y|x)} < v(x', z, y) \frac{f_{Y_1}(y|z)}{1 - F_{Y_1}(y|z)}$$

By *Fact 3*,  $F_{Y_1}(y|x) \geq F_{Y_1}(y|z)$  and thus

$$v(x', x, y) f_{Y_1}(y|x) < v(x', z, y) f_{Y_1}(y|z)$$

This proves that  $\Psi_2 > 0$  implies that  $\Phi_2 > 0$ . Similar argument could show that  $\Psi_{12} > 0$  implies  $\Phi_{12} > 0$ .  $\square$

The proof of Lemma 2 is exactly the same as above, so is omitted.

*Proof of Theorem 2*

*Proof.* The proof mimics the proof of Theorem 1. Again the key point is to show  $\frac{\partial \Pi(b, x)}{\partial b} \geq 0$  if  $b = \beta_f(x') \leq \beta_f(x)$ , and  $\frac{\partial \Pi(b, x)}{\partial b} \leq 0$  if  $b = \beta_f(x') \geq \beta_f(x)$ . Assumption 2 and affiliation ensure that it is not profitable for local deviation. It is trivial to verify that  $\beta_s$  is strictly increasing and differentiable.

From the above argument, the equilibrium payoff to a bidder who receives a signal of  $x$  is  $\Pi(\beta_f(x), x) \geq 0$ , and thus it is individually rational for each bidder to participate in the auction. Similarly, it is easy to show that the equilibrium strategy is increasing and differentiable, hence it is indeed a symmetric equilibrium for the first-price all-pay auction with resale.  $\square$

*Proof of Theorem 3*

*Proof.* Let  $e_f(e_s)$  denote the expected payment in equilibrium of the first-price (second-price)

all-pay auction. Then we have

$$\begin{aligned}
e_s(x) &= \int_0^x \beta_s(y) f_{Y_1}(y|x) dy + [1 - F_{Y_1}(x|x)] \beta_s(x) \\
&= \beta_s(x) F_{Y_1}(x|x) - \int_0^x \beta'_s(y) F_{Y_1}(y|x) dy + [1 - F_{Y_1}(x|x)] \beta_s(x) \\
&= \beta_s(x) - \int_0^x \beta'_s(y) F_{Y_1}(y|x) dy \\
&= \int_0^x v(y, y, y) \frac{f_{Y_1}(y|y)}{1 - F_{Y_1}(y|y)} dy + \int_0^x k(y) dy \\
&\quad - \int_0^x [v(y, y, y) \frac{f_{Y_1}(y|y)}{1 - F_{Y_1}(y|y)} + k(y)] F_{Y_1}(y|x) dy \\
&= \int_0^x v(y, y, y) f_{Y_1}(y|y) \left[ \frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)} \right] dy + \int_0^x k(y) [1 - F_{Y_1}(y|x)] dy \\
&= \int_0^x v(y, y, y) f_{Y_1}(y|y) \left[ \frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)} \right] dy + \int_0^x h(y) \left[ \frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)} \right] dy \\
&\geq \int_0^x v(y, y, y) f_{Y_1}(y|y) dy + \int_0^x h(y) dy \\
&= e_f(x)
\end{aligned}$$

The second equality follows from integration by parts; by *Fact 3*, we have that  $F_{Y_1}(y|x)$  is non-increasing in  $x$ , so  $\frac{1 - F_{Y_1}(y|x)}{1 - F_{Y_1}(y|y)} \geq 1$  for  $x \geq y$ . This gives us the last inequality, which completes the proof.  $\square$

#### *Proof of Proposition 2*

*Proof.* See Krishna and Morgan (1997), proof of Proposition 4.  $\square$

#### *Proof of Proposition 2 & 3*

*Proof.* For any given value of  $X_0$ ,  $\tilde{\beta}$  is similar to  $\beta$ , and the proofs of Proposition 1 and 2 mimic the proofs of Theorem 1 and 2, so are omitted.  $\square$

*Proof of Proposition 4*

*Proof.* Consider the first-price all-pay auction with resale. Let  $\tilde{\beta}_f(\cdot, x_0)$  denote the equilibrium strategy conditional on the revealed of seller's private information  $X_0 = x_0$ . Lemma 2 and affiliation ensures that  $\tilde{\beta}_f(\cdot, x_0)$  is increasing in  $x_0$ .

Let  $e^f(x, z)$  denote the expected payment for bidder 1 if he learns his signal as  $z$  but he bids as if it were  $x$ , and  $\tilde{e}^f(x, z) = E[\tilde{\beta}_f(x, X_0) | Y_1 < x, X_1 = z]$ . Affiliation implies that  $\tilde{e}_2^f(x, z) \geq 0$ . Let  $R(x, z)$  denote the expected value of winning. At the equilibrium, it is optimal to choose  $z = x$ , the resulting first-order condition yields

$$e_1^f(z, z) = \tilde{e}_1^f(z, z)$$

Since  $e_2^f(x, z) = 0$ ,  $\tilde{e}_2^f(x, z) \geq 0$ , then according to *linkage principle*, we have  $\tilde{e}^f(z, z) \geq e^f(z, z)$  since  $\tilde{e}^f(0, 0) = e^f(0, 0) = 0$ . Therefore, in the first-price all-pay auction with resale, the initial seller will benefit from publicly disclosing his private signal.

Using similar argument, we can show that the information disclosure by the seller does not decrease, may increase the expected revenue in the second-price all-pay auction with resale. □

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