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Sealed bid auctions with uncertainty averse bidders★

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Summary. Traditional analysis of auctions assumes that each bidder's beliefs about opponents' valuations are represented by a probability measure. Motivated by experimental findings such as the Ellsberg Paradox, this paper examines the consequences of relaxing this assumption in the first and second price sealed bid auctions with independent private values. The multiple priors model of Gilboa and Schmeidler [*Journal of Mathematical Economics*, **18** (1989), 141–153] is adopted specifically to represent the bidders' (and the auctioneer's) preferences. The unique equilibrium bidding strategy in the first price auction is derived. Moreover, under an interesting parametric specialization of the model, it is shown that the first price auction Pareto dominates the second price auction.

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1 Introduction

The subjective expected utility model axiomatized by Savage (1954) and Anscombe and Aumann (1963) has been the most popular model for studying decision making under uncertainty. In this model, the beliefs of a decision maker are represented by a probability measure. However, the descriptive validity of the model has been questioned since Ellsberg (1961) presented his famous example that people typically prefer to bet on drawing a red ball from an urn containing 50 red and black balls each, than from an urn containing 100 red and black balls in unknown proportions. This behavior reflects an aversion to the "Knightian uncertainty" associated with the ambiguous urn. Subsequent experimental studies generally support that people are averse to uncertainty. (See Camerer and Weber (1992) for a

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survey.) Such aversion contradicts the subjective expected utility model. In fact, it is ruled out by any model of preferences in which underlying beliefs are represented by a probability measure. (Machina and Schmeidler (1992) call this property "probabilistic sophistication". In this paper, I reserve the term "Bayesian" for subjective expected utility maximizer.)

Motivated by the Ellsberg Paradox, which demonstrates that there are situations where the information possessed by a decision maker about the states of nature is too vague or too ambiguous to be representable by a probability measure, alternative models of choice under uncertainty have been proposed. For instance, Gilboa and Schmeidler (1989) develop the multiple priors model, in which the beliefs of the decision maker are represented by a set of probability measures. In this model, the decision maker is said to be uncertainty averse if the set of probability measures representing his beliefs is not a singleton.

Although the Ellsberg Paradox only involves a single decision maker facing an exogenously specified environment, it is natural to think that uncertainty aversion is also common in decision making problems where more than one person is involved. Recently, a number of papers have formulated generalizations of Bayesian solution concepts in games by allowing players' preferences to conform to the multiple priors model.[1] However, serious study of the effects of uncertainty aversion in specific market settings has not yet been carried out.

The purpose of this paper is to investigate the theoretical implications of uncertainty aversion for one very important market setting: sealed bid auctions. The basic assumptions that have been frequently employed in the auction literature are (i) risk neutrality, (ii) independence of private valuations and (iii) symmetry of probabilistically sophisticated beliefs (or more precisely, expected utility preferences). It is well recognized that these assumptions may often fail to portray the auction environment accurately. As a result, there have been a number of explorations into the implications of relaxing each of these assumptions. Probabilistic sophistication is a component of assumption (iii) which has been maintained so far. However, the Ellsberg Paradox and subsequent experimental evidence suggest that this requirement may also be unrealistic. This paper studies the consequences of relaxing this requirement in the first and second price sealed bid auctions. I keep assumptions (i) and (ii) unchanged, but weaken (iii) to symmetry of

[1] For normal form games of complete information, Dow and Werlang (1994), Klibanoff (1993) and Lo (1995a, 1996) generalize Nash Equilibrium. Epstein (1997) generalizes rationalizability and *a posteriori equilibrium*. For normal form games of incomplete information, Epstein and Wang (1996) establish the general theoretical justification for the Harsanyi style formulation. Lo (1995b) provides a generalization of Nash Equilibrium in extensive form games. Note that Epstein (1997) and Epstein and Wang (1996) only require players' preferences to satisfy certain regularity conditions. The class of preferences they look at includes the multiple priors model as a special case.

non-probabilistically sophisticated beliefs (or more precisely, preferences representable by the multiple priors model).[2]

The main contribution of the paper is as follows. First, the unique equilibrium bidding strategy in the first price auction is derived. (When bidders are uncertainty averse, bidding one's true valuation remains a dominant strategy in the second price auction.) Second, a welfare comparison between the first and second price auctions is provided. I show that under an interesting parametric specialization of the multiple priors model, the first price auction Pareto dominates the second price auction. This paper therefore provides additional support for the popularity of the first price auction.

The reason for confining attention to sealed bid auctions is that the multiple priors model is a model of static choice. Formal analysis of auctions in which bidders take more than one action would require extending the multiple priors model to the context of sequential choice. This would bring up some controversial issues, such as updating of non-probabilistically sophisticated beliefs, which deserve a separate treatment. However, the results on the welfare comparison between the first and second price auctions in this paper can still be re-interpreted as a comparison between the two commonly used auction formats, namely the first price and English outcry auctions. The reason is that staying in the English auction at every decision point until the price exceeds one's true valuation remains a dominant strategy for a bidder, regardless of how beliefs are updated. Therefore, under uncertainty aversion, the second price and English auctions continue to generate the same level of utility to the auctioneer.

The structure of this paper is as follows. Since I adopt the multiple priors model to represent the bidders' (and the auctioneer's) preferences, section 2 contains a brief review of the model and a discussion of how it is adapted to the context of sealed bid auctions. The unique equilibrium bidding strategy in the first price auction is derived in section 3. A welfare comparison between the first and second price auctions is conducted in section 4. Some concluding remarks are offered in section 5. Formal proof of Propositions 1, 4 and 5 can be found in the appendix.

2 Preliminaries

2.1 Decision making under uncertainty

In this section, I provide a brief explanation of how the multiple priors model generalizes the subjective expected utility model. For any topological space

[2] Salo and Weber (1995) provide an independent study of the first price auction in which bidders' preferences are represented by the Choquet expected utility model (Schmeidler (1989)). According to this model, the beliefs of the decision maker are represented by a capacity or non-additive probability measure. However, Salo and Weber require the capacity to be a monotonic transformation of a probability measure. This restriction implies that preferences are probabilistically sophisticated and therefore uncertainty averse behavior is excluded (see, for instance, Epstein and Le Breton (1993, p. 13)).

Y, adopt the Borel σ-algebra Σ_Y and denote by $M(Y)$ the set of all probability measures over Y.[3] Adopt the weak* topology on the set of all finitely additive probability measures over (Y, Σ_Y) and the induced topology on subsets. Anscombe and Aumann (1963) propose the following framework for studying decision making under uncertainty. Let (Ω, Σ_Ω) be the space of states of nature and (X, Σ_X) the space of outcomes. For the purpose of this paper, I assume that $X = \mathbf{R}$. Let $\mathscr{F}$ be the set of all bounded measurable functions from Ω to $M(X)$. That is, $\mathscr{F}$ is the set of two-stage, horse-race/roulette-wheel acts. The idea is that the subjective uncertainty regarding Ω will resolve first, and depending on how it resolves, the decision maker will get an objective lottery with prizes out of X. The primitive is the decision maker's preference ordering $\succeq$ over $\mathscr{F}$.

In this framework, the distinction between *risk* and *uncertainty* can be formally seen as follows. $f \in \mathscr{F}$ is called a constant act if $f(\omega) = p \;\; \forall \omega \in \Omega$; such an act involves (probabilistic) risk but no uncertainty. $\succeq$ restricted to the set of constant acts can be identified as the decision maker's preference ordering over the set of objective lotteries $M(X)$. For notational simplicity, I also use $p \in M(X)$ to denote the constant act that yields p in every state of the world and $x \in X$ to denote the degenerate probability distribution on x. The decision maker is *risk neutral* (*risk averse*) if for any $p \in M(X)$, $\int_X x \, dp \sim (\succeq) \; p$.

The preference ordering $\succeq$ restricted to $M(X)$ is represented by the *von Neumann-Morgenstern expected utility model* if it is represented by an *affine* function $u : M(X) \to \mathbf{R}$ (von Neumann and Morgenstern (1953)). The decision maker is risk neutral if $u(x) = x$ for all $x \in X$. The preference ordering $\succeq$ over the set of all acts $\mathscr{F}$ is represented by the *subjective expected utility model* if there exists an *affine* function $u : M(X) \to \mathbf{R}$ and a unique finitely additive probability measure μ on Ω such that for all $f, f' \in \mathscr{F}$,

$$f \succeq f' \Leftrightarrow \int_\Omega u \circ f \; d\mu \geq \int_\Omega u \circ f' d\mu.$$

The probability measure μ can be interpreted as representing the beliefs of the decision maker over the state space Ω.

In the multiple priors model, the single prior in the subjective expected utility model is replaced by a closed and convex set of probability measures. The decision maker evaluates an act by computing the minimum expected utility over the probability measures in his set of priors. To be precise, $\succeq$ is represented by the *multiple priors model* if there exists an affine function $u : M(X) \to \mathbf{R}$ and a unique, nonempty, closed and convex set $\triangle$ of finitely additive probability measures on Ω such that for all $f, f' \in \mathscr{F}$,

$$f \succeq f' \Leftrightarrow \min_{\mu \in \triangle} \int_\Omega u \circ f \; d\mu \geq \min_{\mu \in \triangle} \int_\Omega u \circ f' d\mu. \tag{1}$$

[3] The only exception is that when Y is the space of outcomes X, $M(X)$ denotes the set of all probability measures over X with finite supports.

The decision maker is uncertainty averse if $\triangle$ is not a singleton. In common with the subjective expected utility model, preferences over constant acts, than can be identified with objective lotteries over X, are represented by $u(\cdot)$ and thus conform with the von Neumann Morgenstern model.

The difference between the subjective expected utility model and the multiple priors model can be illustrated by a simple example. Suppose $\Omega = \{\omega_1, \omega_2\}$. Consider an act $f \equiv (f(\omega_1), f(\omega_2))$. If the decision maker is Bayesian and his beliefs over Ω are represented by a probability measure μ, the utility of f is

$$\mu(\omega_1)u(f(\omega_1)) + \mu(\omega_2)u(f(\omega_2)).$$

On the other hand, if the decision maker is uncertainty averse with the set of priors

$$\triangle = \{\mu \in M(\{\omega_1, \omega_2\}) \mid \mu_l \leq \mu(\omega_1) \leq \mu_h \text{ with } 0 \leq \mu_l < \mu_h \leq 1\},$$

then the utility of f is

$$\begin{cases} \mu_l u(f(\omega_1)) + (1-\mu_l)u(f(\omega_2)) & \text{if } u(f(\omega_1)) \geq u(f(\omega_2)) \\ \mu_h u(f(\omega_1)) + (1-\mu_h)u(f(\omega_2)) & \text{if } u(f(\omega_1)) \leq u(f(\omega_2)). \end{cases}$$

Note that given any act f with $u(f(\omega_1)) > u(f(\omega_2))$, $(\omega_1, \mu_l; \omega_2, 1-\mu_l)$ can be interpreted as *local probabilistic beliefs* at f in the following sense. There exists an open neighborhood of f such that for any two acts g and h in the neighborhood,

$$g \succeq h \Leftrightarrow \mu_l u(g(\omega_1)) + (1-\mu_l)u(g(\omega_2)) \geq \mu_l u(h(\omega_1)) + (1-\mu_l)u(h(\omega_2)).$$

That is, the individual behaves like an expected utility maximizer in that neighborhood with beliefs represented by $(\omega_1, \mu_l; \omega_2, 1-\mu_l)$. Similarly, $(\omega_1, \mu_h; \omega_2, 1-\mu_h)$ represents the local probabilistic beliefs at f if $u(f(\omega_1)) < u(f(\omega_2))$. Therefore, decision makers with identical multiple priors preference ordering who "consume" different acts may have different local probability measures at those acts.

2.2 *Sealed bid auctions*

This section formally describes the first price auction game with uncertainty averse bidders and the notion of equilibrium. Foundational justification for such a formulation is provided by Epstein and Wang (1996).

For notational simplicity, assume that there are two bidders 1 and 2.[4] Throughout, any statement concerning bidders i and j is intended for all $i = 1, 2$ and $j \neq i$. An auctioneer has one indivisible unit of a good for sale.

[4] Extension to the case of n bidders is straightforward. Let $\triangle$ be a closed and convex set of probability measures on $[\underline{v}, \overline{v}]$. Define the beliefs of each bidder about the other $n-1$ bidders' valuations to be the closed convex hull of $\{\overbrace{\mu \times \ldots \times \mu}^{n-1 \text{ times}} \mid \mu \in \triangle\}$. This has the interpretation that each bidder believes that the valuation of every bidder is governed by the same probability measure $\mu \in \triangle$, but that he does not know which one. All the results in this paper continue to hold.

Assume that both the bidders and the auctioneer are risk neutral. A reservation price r is imposed and bidders submit sealed bids simultaneously. All bids below r are rejected. The bidder who submits the highest bid wins and pays the price he bids to the auctioneer. If two bidders bid the same amount, each bidder gets the good with equal probability. The following notation is also needed.

- The set of types of bidder i is the set of possible valuations $[\underline{v}, \overline{v}] \subset [0, \infty)$ with typical element v_i. Assume that $r < \overline{v}$.
- The set of actions of bidder i is the set of possible bids $q \cup [r, \infty)$ with typical element b_i. q is a number less than r. It represents the action of not participating in the auction.
- The payoff function $g_i : [\underline{v}, \overline{v}] \times q \cup [r, \infty) \times q \cup [r, \infty) \to M(\mathbf{R})$ of bidder i is defined as

$$g_i(v_i, b_i, b_j) = \begin{cases} v_i - b_i & \text{if } \; b_i > \; b_j \text{ and } b_i \geq r \\ (0, \frac{1}{2}; v_i - b_i, \frac{1}{2}) & \text{if } \; b_i = b_j \geq r \\ 0 & \text{if } \; b_i < b_j \text{ or } b_i = q. \end{cases}$$

  That is, i receives $g_i(v_i, b_i, b_j)$ if his valuation is v_i, if he bids b_i and bidder j bids b_j.
- A *bidding strategy* of bidder i is a measurable function $s_i : [\underline{v}, \overline{v}] \to q \cup [r, \infty)$. This has the interpretation that when i's valuation is v_i, he bids $s_i(v_i)$.

Next, I describe how choosing a bid in the above game can be viewed as a decision problem under uncertainty. Assume that bidder i knows his own valuation v_i and bidder j's bidding strategy s_j, but not j's valuation. The space of uncertainty for bidder i can therefore be regarded as the set of possible valuations $[\underline{v}, \overline{v}]$ of bidder j. Consistent with the multiple priors model, the beliefs of bidder i over the state space $[\underline{v}, \overline{v}]$ are represented by a closed and convex set of probability measures $\triangle$. Bidder i's valuation does not convey any information about the valuation of bidder j and therefore $\triangle$ is independent of v_i. A bid b_i for i corresponds to the act $g_i(v_i, b_i, s_j(\cdot))$ whereby, if the true state is v_j, then i receives the outcome $g_i(v_i, b_i, s_j(v_j))$. Since i can never be better off by bidding above his own valuation, the objective of bidder i is to choose a bid $b_i \leq v_i$ to maximize

$$\min_{\mu \in \triangle} \int_{\underline{v}}^{\overline{v}} g_i(v_i, b_i, s_j(v_j)) \; d\mu(v_j) =$$

$$\begin{cases} \min_{\mu \in \triangle} [v_i - b_i][\mu(\{v_j \mid s_j(v_j) < b_i\}) + \frac{1}{2}\mu(\{v_j \mid s_j(v_j) = b_i\})] & \text{if } b_i \geq r \\ 0 & \text{if } b_i = q. \end{cases}$$

I now turn to the definition of equilibrium. A pair of bidding strategies $\{s_i, s_j\}$ is an *equilibrium* if for all $v_i \in [\underline{v}, \overline{v}]$,

$$\min_{\mu \in \triangle} \int_{\underline{v}}^{\overline{v}} g_i(v_i, s_i(v_i), s_j(v_j)) \; d\mu(v_j) \geq \min_{\mu \in \triangle} \int_{\underline{v}}^{\overline{v}} g_i(v_i, b_i, s_j(v_j)) \; d\mu(v_j) \quad \forall b_i \leq v_i.$$

In words, $\{s_i, s_j\}$ is an equilibrium if it is optimal for bidder i with valuation v_i to bid $s_i(v_i)$, given that j uses the bidding strategy s_j. When $\triangle$ is a singleton, this equilibrium notion collapses to Bayesian Nash Equilibrium.

In this paper, I impose a technical assumption on $\triangle$: every probability measure in $\triangle$ has a strictly increasing and continuous distribution function. Let $\triangle_D$ be the set of distribution functions corresponding to $\triangle$. Whenever it is more convenient, I will refer to $\triangle_D$ as the beliefs of bidder i.

In the second price auction, the bidder who submits the highest bid wins but he is only required to pay a price equal to the second highest bid (or r if no other bidder bids above r). If bidders bid the same amount, each receives the good with equal probability. The second price auction can also be similarly described as a game of incomplete information. This formality is omitted here.

3 Equilibrium

It is evident that when bidders are uncertainty averse, bidding one's true valuation remains a dominant strategy in the second price auction. The purpose of this section is to provide the intuition behind the derivation of the equilibrium of the first price auction game.

Recall that $\triangle_D$ is the set of strictly increasing and continuous distribution functions corresponding to a closed and convex set of probability measures $\triangle$ on $[\underline{v}, \overline{v}]$. Let me begin by defining $F^{\min}$ to be the lower envelope created by $\triangle_D$:

$$F^{\min}(v) = \min_{F \in \triangle_D} F(v) \quad \forall v \in [\underline{v}, \overline{v}]. \tag{2}$$

Note that $\min_{F \in \triangle_D} F(v)$ exists for every $v \in [\underline{v}, \overline{v}]$. The reason is that under the weak* topology, $\triangle$ is compact and $\mu \mapsto \mu([\underline{v}, v])$ is continuous. It is readily verified that $F^{\min}$ is a strictly increasing and continuous probability distribution. Moreover, it first order stochastically dominates every element in $\triangle_D$. Let $\mu^{\min}$ be the probability measure corresponding to $F^{\min}$.

Suppose bidder i with valuation $v_i > r$ knows that bidder j uses a strictly increasing and continuous bidding strategy s_j. Then i knows that the event of having a tie with j is of measure zero and therefore every bid $b_i < v_i$ corresponds to a binary act with the following two outcomes:[5]

$$g_i(v_i, b_i, s_j(v_j)) = \begin{cases} v_i - b_i > 0 & \text{if } v_j < s_j^{-1}(b_i) \\ 0 & \text{if } v_j > s_j^{-1}(b_i). \end{cases}$$

Therefore, bidder i's utility of bidding $b_i \leq v_i$ is equal to

$$\begin{aligned} \min_{\mu \in \triangle} \int_{\underline{v}}^{\overline{v}} g_i(v_i, b_i, s_j(v_j))\, d\mu(v_j) &= \min_{F \in \triangle_D} [v_i - b_i] F(s_j^{-1}(b_i)) \\ &= [v_i - b_i] \min_{F \in \triangle_D} F(s_j^{-1}(b_i)) \end{aligned}$$

[5] I assume that b_i is in the range of s_j. Otherwise, bidder i will simply win or lose with certainty.

$$= [v_i - b_i]F^{\min}(s_j^{-1}(b_i))$$
$$= \int_{\underline{v}}^{\overline{v}} g_i(v_i, b_i, s_j(v_j))\ d\mu^{\min}(v_j).$$

That is, if bidder i knows that j uses a strictly increasing and continuous bidding strategy s_j, then i's preference ordering restricted to the set of bids can be represented by the expected utility function $b_i \mapsto \int_{\underline{v}}^{\overline{v}} g_i(v_i, b_i, s_j(v_j))\ d\mu^{\min}(v_j)$.

If bidders were Bayesians with beliefs represented $F^{\min}$, the unique equilibrium bidding strategy of each bidder is given by

$$s^{\min}(v_i) = v_i - \frac{\int_r^{v_i} F^{\min}(v)\ dv}{F^{\min}(v_i)} \quad \forall v_i > r. \tag{3}$$

(See, for instance, Riley and Samuelson (1981) on how $s^{\min}$ is derived. For uniqueness, see Maskin and Riley (1994).)

Returning to the context where bidders are uncertainty averse with beliefs represented by $\triangle_D$. Now suppose that bidder i knows that bidder j is using the strategy $s^{\min}$. Since $s^{\min}$ is strictly increasing and continuous, bidder i behaves as if he were Bayesian with beliefs represented by $F^{\min}$. Since $s^{\min}$ is the equilibrium bidding strategy corresponding to $F^{\min}$, it is optimal for i with valuation v_i to bid $s^{\min}(v_i)$. This establishes that $s^{\min}$ is an equilibrium bidding strategy when bidders' beliefs are represented by $\triangle_D$. In fact, I prove in the appendix that it is the only equilibrium bidding strategy.

Proposition 1. *In the first price auction,* $\{s^{\min}, s^{\min}\}$ *is the unique equilibrium.*

4 Welfare considerations

It is well known that under assumptions (i), (ii) and (iii) listed in section 1, the following sharp conclusion is obtained:

Revenue Equivalence Theorem. The first and second price auctions yield exactly the same expected profit for every bidder valuation and the same expected revenue to the seller.

In this section, I examine the direction in which the Revenue Equivalence Theorem is altered when we allow preferences conforming to the multiple priors model. Before I proceed, let me clarify the nature of the welfare comparison. For instance, say that the bidders prefer the first price auction rather than the second price auction if the certainty equivalent of participating in the former is higher. Similar meaning is given to the statement that the auctioneer prefers to use the first price auction.

Let us first consider the case where no further restriction is imposed on $\triangle$ and the reservation price r is the same in the first and second price auctions. Since $s^{\min}$ is the unique equilibrium bidding strategy in the first price auction, the utility of bidder i with valuation $v_i > r$ is equal to

$$\min_{\mu\in\triangle}\int_{\underline{v}}^{\overline{v}} g_i(v_i,s^{\min}(v_i),s^{\min}(v_j))\ d\mu(v_j) = \min_{F\in\triangle_D}[v_i - s^{\min}(v_i)]F(v_i).$$

Although beliefs do not affect bidding behavior in the second price auction, they are relevant in determining bidders' welfare. Bidding the true valuation in the second price auction implies that the utility of bidder i with valuation $v_i > r$ is equal to

$$\min_{F\in\triangle_D}\left\{[v_i - r]F(r) + \int_r^{v_i}[v_i - v_j]dF(v_j)\right\}.$$

Given the above, it is straightforward to establish the following.

Proposition 2. *Suppose the reservation price r is the same in the first and second price auctions. Then bidder i with valuation $v_i > r$ (weakly) prefers to participate in the second price auction.*

Proof. For all $v_i > r$,

bidder i's utility in the second price auction

$$= \min_{F\in\triangle_D}\left\{[v_i - r]F(r) + \int_r^{v_i}[v_i - v_j]\ dF(v_j)\right\} \tag{4}$$

$$\geq [v_i - r]F^{\min}(r) + \int_r^{v_i}[v_i - v_j]\ dF^{\min}(v_j) \tag{5}$$

$$= [v_i - s^{\min}(v_i)]F^{\min}(v_i) \tag{6}$$

$$= \min_{F\in\triangle_D}[v_i - s^{\min}(v_i)]F(v_i) \tag{7}$$

$=$ bidder i's utility in the first price auction.

The weak inequality between (4) and (5) follows from the fact that $F^{\min}$ first order stochastically dominates all elements in $\triangle_D$ and the function $\phi(v_j) = \begin{cases} v_i - r & \text{if } v_j \leq r \\ v_i - v_j & \text{otherwise} \end{cases}$ is decreasing in v_j. The equality between (5) and (6) is due to the Revenue Equivalence Theorem. □

Starting from this point, it is important to observe that $F^{\min}$ may or may not be an element of $\triangle_D$. For instance, let F^1 and F^2 be two different probability distributions on $[\underline{v}, \overline{v}]$ and let $\triangle_D =$ convex hull of $\{F^1, F^2\}$. In this case, $F^{\min}$ is the lower envelope created by F^1 and F^2. In Figure 1, $F^{\min}$ is equal to F^1. Therefore, $F^{\min} \in \triangle_D$. In Figure 2, $F^{\min} \notin \triangle_D$.

The significance of the above observation is as follows. If $F^{\min} \notin \triangle_D$, the weak inequality between (4) and (5) can be strict. On the other hand, if $F^{\min} \in \triangle_D$, the weak inequality will become an equality and bidder i will be indifferent between the two auction formats. Note that the equality between (5) and (6) is under the assumption that the reservation price is the same in the two auctions. Since for all $v_i > r$, $s^{\min}(v_i)$ increases as r increases, the expression in (5) will be less than that in (6) if the reservation price in the first

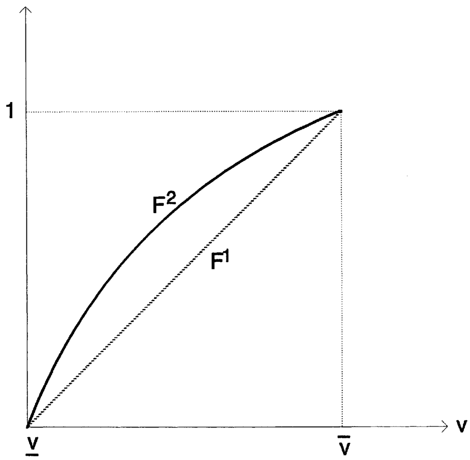

Figure 1. $F^{\min} \in \triangle_D$

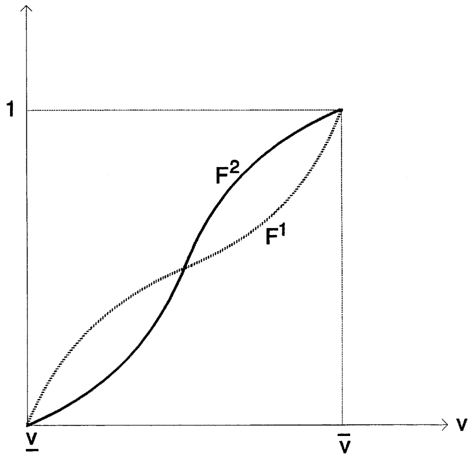

Figure 2. $F^{\min} \notin \triangle_D$

price auction is lower. This constitutes a proof of Proposition 3. The result will be useful at the end of this section.

Proposition 3. *Suppose $F^{\min} \in \triangle_D$ and the reservation price r in the first price auction is strictly less than the reservation price in the second price auction. Then bidder i with valuation $v_i > r$ strictly prefers to participate in the first price auction.*

Next, consider the following parametric specialization of $\triangle$. Let μ^* be a particular probability measure on $[\underline{v}, \overline{v}]$ which possesses a strictly increasing and continuous distribution function. Define the following closed and convex set of probability measures:

$$\triangle^{l,L} = \{\mu \in M([\underline{v}, \overline{v}]) \mid \mu \ll \mu^* \;\; \text{and} \;\; l \leq \frac{d\mu}{d\mu^*} \leq L\}. \tag{8}$$

The above specification is intuitive and self-explanatory. μ^* may be thought of as the "true" probability law governing each bidder's valuation. However the bidders are vague about this true law and have beliefs represented by $\triangle^{l,L}$. The absolute continuity requirement for all perturbations indicates a lack of vagueness about null events. It is easy to see that $\triangle^{l,L} = \{\mu^*\}$ if $l = L = 1$, corresponding to the situation where the bidders know the probability law μ^*. When $l < 1$ and $L > 1$, $\triangle^{l,L}$ contains more than one probability measure and $\mu^* \in \triangle^{l,L}$. Henceforth, it is assumed that $l < 1 < L$. Finally, note that $\mu^{\min} \in \triangle^{l,L}$ (or $F^{\min} \in \triangle_D^{l,L}$).

So far, I have not mentioned the preferences of the auctioneer. Note that the relevant state space for the auctioneer is $[\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}]$ which represents the set of possible valuations of bidders i and j. The choice of an auction format corresponds to choosing an act over this state space. Assume that the auctioneer's preferences are also represented by the multiple priors model defined in (1). A specification of the auctioneer's beliefs which parallels (8) is as follows:

$$\triangle^{\mathscr{A}} = \text{closed convex hull of } \{\mu \times \mu \in M([\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}]) \mid \mu \in \triangle^{l,L}\}. \tag{9}$$

The probability measures in $\{\mu \times \mu \in M([\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}]) \mid \mu \in \triangle^{l,L}\}$ takes the form of $\mu \times \mu$ because the auctioneer believes that the valuation of each bidder is governed by the same probability measure $\mu \in \triangle^{l,L}$, but that he does not know which one.

When the bidders' and the auctioneer's beliefs are represented by $\triangle^{l,L}$ and $\triangle^{\mathscr{A}}$ respectively, they are both uncertainty averse. Moreover, their beliefs "agree" in the sense that

$$\triangle^{l,L} = \big\{\mu \in M([\underline{v}, \overline{v}] \mid \exists \bar{\mu} \in \triangle^{\mathscr{A}} \\ \text{such that } \mu \text{ is the marginal probability measure derived from } \bar{\mu}\big\}.$$

That is, the set of marginal probability measures on $[\underline{v}, \overline{v}]$ as one varies over all the probability measures in $\triangle^{\mathscr{A}}$ is equal to $\triangle^{l,L}$. This represents the counterpart of the Bayesian scenario where the bidders' and the auctioneer's beliefs are represented by μ^* and $\mu^* \times \mu^*$ respectively. I also examine some environments in which the bidders and the auctioneer have asymmetric beliefs. Consider, therefore, the following three scenarios:

1. The bidders' (auctioneer's) beliefs are represented by $\triangle^{l,L}$ ($\triangle^{\mathscr{A}}$).
2. The bidders' (auctioneer's) beliefs are represented by μ^* ($\triangle^{\mathscr{A}}$).
3. The bidders' (auctioneer's) beliefs are represented by $\triangle^{l,L}$ ($\mu^* \times \mu^*$).

2 and 3 represent the situations where the true probability law is known to the bidders and the auctioneer respectively.

Proposition 4. *Suppose beliefs satisfy either* 1, 2 *or* 3 *and the reservation price r is the same in the first and second price auctions. Then the auctioneer strictly prefers to use the first price auction.*

Let me provide a summary of the proof of Proposition 4. Suppose the bidders' and auctioneer's beliefs are represented by $\triangle^{l,L}$ and $\triangle^{\mathscr{A}}$ respectively. As explained in section 2.1, although the global beliefs of the bidders and auctioneer agree, their local beliefs may disagree when they are consuming different acts. When the auctioneer is considering which auction format to adopt, uncertainty aversion leads him to think that it is likely that the bidders have low valuations. There exists a probability measure $\mu^H \in \triangle^{l,L}$ such that μ^H is first order stochastically dominated by all elements in $\triangle^{l,L}$. The auctioneer behaves as if his beliefs over a bidder's valuation were represented by μ^H. Recall that the bidders always bid their true valuation in the second price auction. Therefore, the Revenue Equivalence Theorem implies that the auctioneer would be indifferent between the two auction formats if the bidders were using the bidding strategy s^H in the first price auction where $s^H(v_i) = v_i - \frac{\int_r^{v_i} H(v)dv}{H(v_i)}$ $\forall v_i > r$. On the other hand, when the bidders are considering the optimal bid in the first price auction, uncertainty aversion leads them to think that it is likely that their opponents have high valuation. Recall that $\mu^{\min} \in \triangle^{l,L}$ first order stochastically dominates all elements in $\triangle^{l,L}$. The bidders behave as if their beliefs were represented by $\mu^{\min}$. As a result, the bidders are actually using the bidding strategy $s^{\min}$ defined in (3). Then I prove that $s^{\min}(v_i) \geq s^H(v_i)$ for all $v_i > r$, where the dominance is strict over an interval.[6] This is the reason that the auctioneer strictly prefers to use the first price auction. The proof for the other two cases is similar.

Proposition 4 says that the auctioneer's utility in the first price auction is strictly higher if he sets the same reservation price in the two auction formats. This implies that the auctioneer also strictly prefers to use the first price auction if he has the freedom to set the reservation price optimally in each auction format. In fact, under the above specifications of beliefs, the reservation price that maximizes the auctioneer's utility in the first price auction is strictly lower than that in the second price auction.

Proposition 5. *Suppose beliefs satisfy either* 1, 2 *or* 3. *Then the optimal reservation price in the first price auction is strictly lower than that in the second price auction.*

Proposition 5 and the fact that $\mu^{\min} \in \triangle^{l,L}$ enable us to invoke Proposition 3 to conclude that the bidders also strictly prefer the first price auction. Moreover, if the optimal reservation price in the second price auction is above $\underline{v}$, Proposition 5 implies that the first price auction is more efficient in the sense that the set of states in which a sale occurs in the first price auction strictly includes that in the second price auction.

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[6] The fact that $\mu^{\min}$ first order stochastically dominates μ^H is not sufficient to imply that $s^{\min}$ dominates s^H.

5 Concluding remarks

Experimental findings inspired by the Ellsberg Paradox demonstrate that people are typically averse to uncertainty. This motivates the development of the multiple priors model. Recently, this model has been adopted to generalize various Bayesian solution concepts in games. However, to demonstrate that the generalized equilibrium concepts are worth taking seriously by economists, we have to show that they make a difference in real economic situations. This paper is a first step in this research direction. Let me evaluate its contribution by addressing the following question: Do the results in this paper enable us to conclude that uncertainty aversion makes a difference in the context of first and second price sealed bid auctions?

The principle result of this paper is that under an interesting parametric specialization of the multiple priors model, the first price auction Pareto dominates the second price auction. Since similar results on the welfare comparison between the two auction formats have been obtained by assuming that bidders are Bayesians but risk averse, it may lead the reader to think that uncertainty aversion is just a different way to model "cautious" behavior, adding nothing new in terms of results. Therefore, it is worth explaining that given the existence of the auction literature with risk averse bidders, the value added of considering uncertainty aversion is still significant. Although my argument does not provide a definitive answer, it is sufficiently positive to encourage future research.

First, the similar results on the welfare comparison produced by risk aversion and uncertainty aversion should be viewed as complementary rather than mutually exclusive. When bidders are risk averse, the auctioneer strictly prefers using the first price rather than the second price auction. (See Riley and Samuelson (1981, p. 388, Proposition 4). For more general results, see Maskin and Riley (1984, pp. 1489–1490, Theorems 4 and 5).) In this paper, I show that the auctioneer also strictly prefers using the first price auction when bidders are uncertainty averse with beliefs conforming to the parametric specialization $\triangle^{l,L}$ defined in (8). This can be viewed as an alternative and more attractive explanation for the use of the first price auction in situations where risk aversion is not an important consideration.

There is another important instance where risk aversion and uncertainty aversion may play a complementary role. In the first price auction, as bidders become more risk averse, they make uniformly higher bids (Riley and Samuelson (1981, p. 388, Proposition 4)). However, it is well recognized that risk aversion can only partially explain the experimental findings that submitted bids are typically higher than what risk neutral bidders would submit in equilibrium (see Kagel (1995, pp. 523–536) for a survey):

> "Results of this dialogue suggest that with respect to the primary issue, the role of risk aversion in first-price auctions, it is probably safe to say that risk aversion is one element, but far from the only element, generating bidding above the RNNE (risk neutral Nash Equilibrium). This is not to say that there is no longer any debate regarding the relative

importance of risk aversion versus other factors, or in terms of what these other factors are." (Kagel (1995, p. 525))

Some results in this paper are preserved when bidders are both uncertainty averse and risk averse with the same *affine* function $u : M(X) \to \mathbf{R}$. In particular, redefine $s^{\min}$ to be the equilibrium bidding strategy corresponding to the case where bidders' risk aversion is represented by u and beliefs represented by $\mu^{\min}$. That is, $s^{\min}$ is the bidding strategy satisfying the following differential equation:

$$\frac{ds^{\min}(v)}{dv} = \frac{\frac{dF^{\min}(v)}{dv}}{F^{\min}(v)} \frac{u(v - s^{\min}(v))}{\frac{du(v - s^{\min}(v))}{dv}}$$

(Riley and Samuelson (1981, p. 391)). The proof of Proposition 1 in this paper goes through exactly as before. Now suppose the bidders' beliefs are represented by $\triangle^{l,L}$ defined in (8). As explained in section 4, the interpretation of $\triangle^{l,L}$ is that there is a true probability measure $\mu^* \in \triangle^{l,L}$ governing each bidder's valuation. However, the bidders are vague about this true law. Similar to the first part of the proof of Proposition 4 in the appendix, it can be shown that $s^{\min}$ dominates the equilibrium bidding strategy when bidders' risk aversion is represented by u and beliefs represented by μ^*. That is, if the bidders are both risk averse and vague about the true probability law μ^*, they will bid even more aggressively than when they are risk averse but certain about μ^*. This may be a better explanation for the experimental findings on bidding above the risk neutral Nash Equilibrium. Since the probability measure μ^* governing each bidder's valuation is typically announced in the auction experiments, this conjecture implicitly assumes that the experimental subjects do not have full confidence about the announcement and therefore they are vague about μ^*. Of course, uncertainty aversion may come from other sources. For instance, the subjects may be uncertain about the strategies of their opponents. This may be due to an inability to know whether their opponents are fully rational, or due to ignorance about opponents' preferences. Investigation of how uncertainty about opponents' rationality and preferences affect bidding behavior is another subject for future research. Needless to say, this is only meant to be some speculation on the role of uncertainty aversion in the context of first price auction. Formal experimental investigation is beyond the scope of this paper. I only intend to emphasize that uncertainty aversion may be an important issue and deserves further attention.

Second, risk aversion and uncertainty aversion produce similar but different results on the welfare comparison between the first and second price auctions. When bidders are risk averse, the auctioneer **always** strictly prefers using the first price rather than the second price auction. In this paper, I establish the same result when bidders are uncertainty averse with beliefs conforming to the parametric specialization $\triangle^{l,L}$ defined in (8). When beliefs deviate from the parametric specialization, examples can be constructed to predict the opposite. This suggests another topic for future research. Is there any interesting parametric specialization of the multiple priors model which predicts that the auctioneer strictly prefers using the second price auction? More generally, is it possible to provide a variety of parametric specialization of beliefs, where each of them is an appropriate representation of the information characteristic of the auction of a particular market? Can this help explain the casual observations that both the first price and the second price auctions are used in environments where risk aversion seems to be significant (see, for instance, McAfee and McMillan (1987, p. 726))? This is a sensible exercise because uncertainty aversion is "endogenous" in the sense of depending on the information structure of the decision problem under consideration. For example, in the setting of the Ellsberg experiment, the same decision maker could be uncertainty averse when he bets on drawing a ball from an urn containing 100 red and black balls in unknown proportions, but probabilistically sophisticated when he bets on drawing a ball from another urn and he is told that the urn contains 50 red and black balls each.

Finally, from the bidders' point of view, risk aversion and uncertainty aversion also produce some different results. Suppose the reservation price is the same in the first and second price auctions. Uncertainty averse bidders **always** prefer the second price auction (Proposition 2 in this paper). However, risk averse bidders may prefer either the first or the second price auction (Matthews (1987, p. 638, Theorem 1)).

To conclude, I would like to point out some general issues that we need to deal with to enhance further application of the multiple priors model to different auction environments. One important issue is existence (and uniqueness) of equilibrium. In this paper, the derivation of the equilibrium bidding strategy in the first price auction depends on the following observation. If bidder j uses a strictly increasing bidding strategy, then any bid submitted by bidder i corresponds to a two outcome act: win (lose) if j's valuation is below (above) a particular number. This observation continues to be applicable to some auction formats. Since auctions are games of incomplete information, it would be useful to extend the work of Balder (1988) and Milgrom and Weber (1985) to provide general results on existence of equilibrium for games of incomplete information with more general players' preferences. Another fundamental issue is the formulation of various concepts in the auction literature, such as affiliation, outside the probability framework. It remains to be seen whether uncertainty aversion will deliver interesting new predictions in more general contexts.

Appendix

Proof of Proposition 1

I first establish that $\{s^{\min}, s^{\min}\}$ is an equilibrium. Given that bidder j uses $s^{\min}$, bidder i with valuation $v_i > r$, if he bids at all, will wish to bid in the range of this function. Hence we can write any bid as $b_i = s^{\min}(v)$ and view bidder i as choosing v. It follows that $\{s^{\min}, s^{\min}\}$ is an equilibrium if bidder i

can do no better than choose $v = v_i$, and so bids $s^{\min}(v_i)$. To prove this, it suffices to observe that for all $s^{\min}(v) \leq v_i$,

$$\min_{F \in \triangle_D} [v_i - s^{\min}(v)]F(v) = [v_i - s^{\min}(v)] \min_{F \in \triangle_D} F(v) \tag{10}$$

$$= [v_i - s^{\min}(v)]F^{\min}(v). \tag{11}$$

(10) is the utility of bidder i if he bids $s^{\min}(v)$. (11) is the utility of bidder i if he bids $s^{\min}(v)$ and if his beliefs were represented by $F^{\min}$. Since $s^{\min}$ is the equilibrium bidding strategy corresponding to $F^{\min}$, (11) must be maximized at $v = v_i$. This completes the proof that $\{s^{\min}, s^{\min}\}$ is also an equilibrium when bidders' beliefs are represented by $\triangle_D$.

Next, I demonstrate that the above equilibrium is unique. Suppose $\{s_i, s_j\}$ is an equilibrium. It can be shown as in the Bayesian case (see, for instance, Fudenberg and Tirole (1991, p. 223)) that s_i and s_j are non-decreasing. Moreover, if bidders i and j bid according to s_i and s_j respectively, the event of having a tie is of measure zero and therefore can be ignored. Recall that $\mu^{\min}$ denotes the probability measure corresponding to $F^{\min}$. The above properties of $\{s_i, s_j\}$ enable us to establish that for all $v_i > r$ and for all $b_i \leq v_i$,

$$[v_i - s_i(v_i)]\mu^{\min}(\{v_j \mid s_j(v_j) < s_i(v_i)\}) \tag{12}$$

$$= [v_i - s_i(v_i)] \min_{\mu \in \triangle} \mu(\{v_j \mid s_j(v_j) < s_i(v_i)\}) \tag{13}$$

$$= \min_{\mu \in \triangle} [v_i - s_i(v_i)]\mu(\{v_j \mid s_j(v_j) < s_i(v_i)\}) \tag{14}$$

$$\geq \min_{\mu \in \triangle} [v_i - b_i][\mu(\{v_j \mid s_j(v_j) < b_i\}) + \frac{1}{2}\mu(\{v_j \mid s_j(v_j) = b_i\})] \tag{15}$$

$$\geq \frac{1}{2}[v_i - b_i]\left[\min_{\mu \in \triangle} \mu(\{v_j \mid s_j(v_j) < b_i\}) + \min_{\mu \in \triangle} \mu(\{v_j \mid s_j(v_j) \leq b_i\})\right] \tag{16}$$

$$= \frac{1}{2}[v_i - b_i][\mu^{\min}(\{v_j \mid s_j(v_j) < b_i\}) + \mu^{\min}(\{v_j \mid s_j(v_j) \leq b_i\})] \tag{17}$$

(12) is bidder i's utility of bidding $s_i(v_i)$ if his beliefs were represented by $F^{\min}$. The weak inequality between (14) and (15) is due to the hypothesis that it is optimal to bid $s_i(v_i)$ when bidder i's beliefs are represented by $\triangle_D$. (17) is bidder i's utility of bidding b_i if his beliefs were represented by $F^{\min}$. The inequalities between (12) and (17) therefore enable us to conclude that $\{s_i, s_j\}$ is also the equilibrium if bidders' beliefs were represented by $F^{\min}$. Therefore, $s_i = s_j = s^{\min}$. □

Proof of proposition 4

The proof for the case where beliefs are represented by $\triangle^{l,L}$ and $\triangle^{\mathscr{A}}$ is provided below. Straightforward modification of the proof will suffice to prove the other two cases. Let F^* be the probability distribution corresponding to μ^*. Given $\triangle^{l,L}$,

$$F^{\min}(v) = \begin{cases} lF^*(v) & \text{if } v \le \tilde{v} \\ LF^*(v) - L + 1 & \text{otherwise} \end{cases}$$

where $\tilde{v}$ satisfies $F^*(\tilde{v}) = \frac{L-1}{L-l}$. Recall that $s^{\min}$ defined in (3) is the equilibrium bidding strategy when bidders' beliefs are represented by $\triangle^{l,L}$.

Define the probability distribution H as follows:

$$H(v) = \begin{cases} LF^*(v) & \text{if } v \le \hat{v} \\ lF^*(v) - l + 1 & \text{otherwise} \end{cases}$$

where $\hat{v}$ is the value which satisfies $F^*(\hat{v}) = \frac{1-l}{L-l}$. Algebraic manipulation leads to the following relationship between $F^{\min}$ and H:

$$\frac{H(x)}{H(v)} \begin{cases} = \frac{F^{\min}(x)}{F^{\min}(v)} & \text{if } x \in [\underline{v}, v] \text{ and } v \le \min\{\hat{v}, \tilde{v}\} \\ > \frac{F^{\min}(x)}{F^{\min}(v)} & \text{if } x \in [\underline{v}, v) \text{ and } v > \min\{\hat{v}, \tilde{v}\}. \end{cases} \tag{18}$$

Recall that s^H denotes the equilibrium bidding strategy if bidders' beliefs were represented by H. Note that $s^{\min}$ can be rewritten (using integration by parts) as

$$s^{\min}(v) = \frac{1}{F^{\min}(v)} \left[rF^{\min}(r) + \int_r^v x \, dF^{\min}(x) \right] \quad \forall v > r \, ,$$

and similarly for s^H. That is, $s^{\min}(v)$ is the expected value of the function ϕ with respect to the probability distribution $\frac{F^{\min}(\cdot)}{F^{\min}(v)}$ where

$$\phi(x) = \begin{cases} r & \text{if } x \le r \\ x & \text{if } x > r \, , \end{cases}$$

and similarly for $s^H(v)$. Since ϕ is increasing and strictly increasing when $x > r$, $s^{\min}(v) > s^H(v)$ for any $v > r$ if $\frac{F^{\min}(\cdot)}{F^{\min}(v)}$ first order stochastically dominates $\frac{H(\cdot)}{H(v)}$. Therefore, (18) implies that

$$s^{\min}(v) \begin{cases} = s^H(v) & \forall r < v \le \min\{\hat{v}, \tilde{v}\} \\ > s^H(v) & \forall v > \min\{\hat{v}, \tilde{v}\}. \end{cases} \tag{19}$$

(In (19), I assume that $r < \min\{\hat{v}, \tilde{v}\}$. Otherwise, we simply have $s^{\min}(v) > s^H(v)$ for all $v > r$.)

Define the bounded measurable and non-decreasing functions Φ^1, Φ^2 and Φ^3 as follows:

$$\Phi^1(v_i, v_j) = \begin{cases} 0 & \text{if } v_i \le r \text{ and } v_j \le r \\ s^{\min}(v_i) & \text{if } v_i > r \text{ and } v_j \le r \\ s^{\min}(v_j) & \text{if } v_i \le r \text{ and } v_j > r \\ \max\{s^{\min}(v_i), s^{\min}(v_j)\} & \text{if } v_i > r \text{ and } v_j > r, \end{cases}$$

$$\Phi^2(v_i, v_j) = \begin{cases} 0 & \text{if } v_i \le r \text{ and } v_j \le r \\ s^H(v_i) & \text{if } v_i > r \text{ and } v_j \le r \\ s^H(v_j) & \text{if } v_i \le r \text{ and } v_j > r \\ \max\{s^H(v_i), s^H(v_j)\} & \text{if } v_i > r \text{ and } v_j > r, \end{cases}$$

$$\Phi^3(v_i, v_j) = \begin{cases} 0 & \text{if } v_i \le r \text{ and } v_j \le r \\ r & \text{if } v_i > r \text{ and } v_j \le r \text{ or if } v_i \le r \text{ and } v_j > r \\ \min\{v_i, v_j\} & \text{if } v_i > r \text{ and } v_j > r. \end{cases}$$

The next step is to turn to some properties of $\triangle^{l,L}$. Note that the probability measure μ^H corresponding to H is an element of $\triangle^{l,L}$ and it is first order stochastically dominated by all elements in $\triangle^{l,L}$. Therefore, for all $i = 1, 2, 3$,

$$\int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^i(v_i, v_j) d\mu(v_i) d\mu(v_j) \ge \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^i(v_i, v_j) d\mu^H(v_i) d\mu^H(v_j) \quad \forall \mu \in \triangle^{l,L}. \tag{20}$$

According to the definition of closed convex hull, (20) can be extended to

$$\int_{[\underline{v},\overline{v}]\times[\underline{v},\overline{v}]} \Phi^i(v_i, v_j)\ d\bar{\mu}(v_i, v_j) \ge \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^i(v_i, v_j)\ d\mu^H(v_i)\ d\mu^H(v_j) \quad \forall \bar{\mu} \in \triangle^{\mathscr{A}}. \tag{21}$$

I am now in a position to establish that

the auctioneer's utility in the first price auction

$$= \min_{\bar{\mu}\in\triangle^{\mathscr{A}}} \int_{[\underline{v},\overline{v}]\times[\underline{v},\overline{v}]} \Phi^1(v_i, v_j)\ d\bar{\mu}(v_i, v_j) \tag{22}$$

$$= \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^1(v_i, v_j)\ d\mu^H(v_i)\ d\mu^H(v_j) \tag{23}$$

$$> \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^2(v_i, v_j)\ d\mu^H(v_i)\ d\mu^H(v_j) \tag{24}$$

$$= \int_{\underline{v}}^{\overline{v}}\int_{\underline{v}}^{\overline{v}} \Phi^3(v_i, v_j)\ d\mu^H(v_i)\ d\mu^H(v_j) \tag{25}$$

$$= \min_{\bar{\mu}\in\triangle^{\mathscr{A}}} \int_{[\underline{v},\overline{v}]\times[\underline{v},\overline{v}]} \Phi^3(v_i, v_j)\ d\bar{\mu}(v_i, v_j) \tag{26}$$

= the auctioneer's utility in the second price auction.

(21) implies the equality between (22) and (23) and that between (25) and (26). (19) implies the inequality between (23) and (24). The equality between (24) and (25) is again due to the Revenue Equivalence Theorem. □

Proof of Proposition 5

Again, only the proof for the case where beliefs are represented by $\triangle^{l,L}$ and $\triangle^{\mathscr{A}}$ is presented below. According to the proof of Proposition 4, the auctioneer's net advantage (in terms of utility) obtained from using the first rather than the second price auction is equal to the difference between the expressions in (23) and (24), which can be rewritten as

$$2\int_r^{\bar{v}} [s^{\min}(v) - s^H(v)]H(v)\; dH(v). \tag{27}$$

Differentiating (27) with respect to r, we have

$$2\int_r^{\bar{v}} \left[\frac{\partial s^{\min}(v)}{\partial r} - \frac{\partial s^H(v)}{\partial r}\right] H(v)\; dH(v). \tag{28}$$

To prove Proposition 5, it suffices to show that the expression in (28) is negative. For all $v > r$,

$$\frac{\partial s^{\min}(v)}{\partial r} - \frac{\partial s^H(v)}{\partial r} = \frac{F^{\min}(r)}{F^{\min}(v)} - \frac{H(r)}{H(v)}.$$

Therefore, we can use (18) to conclude the proof. □

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