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Economic Theory

Auctions with cross-shareholdings*

Sudipto Dasgupta[1] **and Kevin Tsui**[2]

[1] Department of Finance, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, HONG KONG (e-mail: dasgupta@ust.hk)
[2] Department of Economics, University of Chicago, Chicago, IL 60637-2799, USA (e-mail: ktsui@uchicago.edu)

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Summary. We study the effect of cross-shareholding among two competing firms on their bidding behavior and the expected sales revenue for the seller in an auction environment. The bidders' private signals are independent, and the model encompasses the private values model and a particular common value model as special cases. When cross-shareholding is symmetric, the bids decrease towards the collusive level as the degree of cross-shareholding increases. The Revenue Equivalence result no longer holds: the first-price auction generates higher expected revenue for the seller than the second-price auction.With asymmetric cross-shareholding, revenue comparisons are only possible in the common value setting. Expected revenue for the seller is again higher in the first-price than in the second price auction. Bidding behavior in the second-price auction is more sensitive to changes in cross-shareholding and the value environment than in the first-price auction.

Keywords and Phrases: Cross-shareholdings, Auctions, Bidding, Revenue equivalence.

JEL Classification Numbers: C72, D44.

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Correspondence to: S. Dasgupta

1 Introduction

Competing firms may often hold shares in each other. In this paper, we analyze the effect of such cross-shareholding when the firms bid against each other in an auction. When two bidders have *symmetric* cross-holdings, the firms' bids are decreasing in the common degree of cross-ownership.[1] More interestingly, the seller's expected sales revenue is no longer independent of the auction mechanism. In a fairly general specification that encompasses the private values and a particular common value model as special cases, we show that the seller prefers the first-price auction to the second-price auction for symmetric positive cross-shareholding. The optimal reserve price set by the seller is also higher in the second-price auction than in the first-price auction, indicating that there is a lower probability of trade under the former.

When asymmetric cross-shareholding is introduced, revenue comparisons are only possible for the common value model, where the first price auction is once again shown to generate higher expected sales revenue than the second-price auction. One of the main implications of the analysis is that bidding behavior under the first-price auction is more robust to a change in the underlying value environment and to asymmetries in cross-holdings than in the second-price auction. For example, an interesting – and counter-intuitive – result concerns the bidding behavior of asymmetric firms under the two alternative auction forms. In the first-price auction, irrespective of whether a private or common value model is considered, the bidder with the higher cross-shareholding bids less aggressively.[2] For the second-price auction, however, the result is sensitive to the particular value environment being considered within our general setting. For a special case of the private values model, we are able to show that the bidder with the higher cross-shareholding does bid less aggressively, and we can also show that for the private values model in general, there is always some (common) range of realizations of the bidders' private signals over which the bidder with larger cross-shareholding bids less aggressively. However, surprisingly, for the common value case, the bidder with larger cross-shareholding bids *more* aggressively. Further, within the common value setting, the bidding behavior is more sensitive to changes in cross-shareholding in the second-price auction than in the first-price auction.

The analysis of bidding competition when competing firms may hold shares in each other seems empirically relevant. In many countries, one of the primary categories of block shareholders in a company is "other corporations". For example,

[1] The "degree of cross-ownership" for a bidder is the ratio of his ownership share in the other firm to his ownership share in his own firm (i.e. the firm he controls). An increase in the degree of cross-ownership can be associated with either an increase in the fraction of shares of the rival firm that is owned, or a decrease in the fraction of shares in the own firm. We will refer to a bidder with a higher degree of cross-ownership as simply a bidder with higher cross-ownership or cross-shareholding.

[2] A bidder is said to bid less (more) aggressively than its rival if, for the same realization of the private signal, that bidder bids lower (higher) than the rival, for all values of the private signal. A similar definition applies with respect to a change in the bidding behavior of a bidder as a model parameter changes.

in a study of immediate ownership of companies in nine Asian countries[3], Claessens et al. [3] report that other corporations as a group were the single most important ownership category in all nine countries, with a mean ownership percentage ranging from 7.2% for Taiwan to 52.5% for Indonesia. A significant part of this ownership, no doubt, is accounted for by companies that are part of the same "business group" such as the *Keiretsu* in Japan and *Chaebols* in Korea, which may not be competitors. However, even for independent (non-group affiliated) companies, other companies constitute a significant percentage of the ownership.[4] While relatively little is known about the relationships that these other corporate bodies have with the firms in which they hold shares, when they are not part of the same business group, it is likely that the firms might be related vertically (e.g. a buyer might hold shares of a supplier firm) as well as horizontally (i.e. they might compete for contracts, or market share).

Hansen and Lott [8] argue that when shareholders are diversified, and own shares in firms that may impose externalities on each other, a similar "internalization" of the externalities that exists when firms directly own shares in each other is likely to occur. They point out that institutional investors such as pension and mutual funds in the U.S. own diversified portfolios and often hold shares in competing firms in an industry. For example, they find that in the U.S. computer industry in 1994, the ownership share of institutions that held shares in two or more of the top six companies ranged from 36.5% (Microsoft) to 77.4 % (intel) in these companies. For the big three companies in the automobile industry, these numbers ranged from 35.6% (GM) to 56% (Chrysler). If we agree with Hansen and Lott [8] that such internalization does occur, then our model applies not just to cases of direct cross-shareholdings between firms but much more widely.[5]

The rest of the paper is organized as follows. In Section 2, we discuss how our model relates to existing literature. In Section 3, we outline the general model which encompasses both private and a particular case of the common value model as special cases. In Section 4, we analyze bidding behavior under symmetric cross-shareholding, while in Section 5, we introduce asymmetric cross-shareholding. In Section 6, we specialize the model to the common values framework as in Bulow, Huang and Klemperer [1] to investigate the role of asymmetries and the value environment in greater detail. Finally, Section 7 concludes.

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[3] The countries are Hong Kong, Indonesia, Japan, Soth Korea, Malayasia, Philippines, Singapors, Taiwan and Thailand.

[4] For example, in India, where group affiliation is relatively easy to identify, ownership by other corporate bodies (excluding financial institutions and mutual funds) accounts for about 20% of the ownership, on average. In contrast, for companies that belong to one of the top fifty business groups (identified on the basis of size), other corporations account for about a third of the ownership.

[5] For example, suppose a third party owns a quarter of the shares in both firms, labelled 1 and 2. Then firm 1 may be assumed to maximize the sum of profits of shareholders who only own shares in firm 1 and the third party who owns shares in both firms. This is equivalent to maximizing an objective function equal to the sum of own profits plus a quarter of the other firm's profits. If the Hansen-Lott argument about internalization of externalities is important, policy implications emerge from our analysis. For example, if firms competing for government contracts internalize the externalities, then our analysis suggests that the sealed bid auction may be a better method to sell government contracts than the open auction.

2 Related literature

Our analysis is related to at least four strands of literature. In a number of recent papers, Jehiel and Moldovanu [10,11] and Jehiel, Moldovanu and Stacchetti [12,13] analyze the effect of externalities that the sale of an object to a particular bidder imposes on other bidders. Considerations of externalities are relevant in a number of practical situations, such as the sale of an innovation to competing oligopolistic firms, mergers between downstream firms where synergies are involved, and the effect that the possible sale of nuclear weapons by countries in the former Soviet Union to certain other countries may have on the remaining nuclear superpowers, i.e. the United States and Russia. In the papers cited above, the authors are interested in two main questions in the presence of externalities: the design of an optimal selling mechanism and the allocative properties of that mechanism, and the effect of reserve prices and entry fees in a standard auction such as the second-price auction (Jehiel and Moldovanu, [11]). Since in our context, a losing bidder gains some surplus if the object is allocated to the rival, it can be thought of as a "positive externality". However, there are important differences between our framework and the ones considered in these papers. These differences stem from the fact that the "positive externality" in our case not only depends on the rival's private value (as in the above papers), but also the winning price that the rival must pay. In the first price auction, this winning price is a also a function of the rival's private value, but in the second price auction, it is a function of own private value (since the winning rival must pay the loser's bid). Thus, the cases of the first-price and the second-price auction belong to different *classes* of mechanisms (see Jehiel, Moldovanu and Stacchetti [12,13]), and a comparison seems useful.[6]

One way in which the difference manifests itself is that in Jehiel and Moldovanu [11], in the presence of a binding reserve price set by the seller, there is no symmetric separating equilibrium in pure strategies in the second price auction (the only auction mechanism they consider) for the case of positive externalities. The reason is as follows. Consider a bidder whose private value is slightly above that of the marginal bidder. This bidder's bid cannot be very close to his value, because then his surplus from winning is almost zero, but by dropping his bid to the reserve price, he will now lose against those types against whom he would have won previously, and get the positive externality as his surplus. Thus, every bidder (including the the marginal bidder) must get a strictly positive surplus (at least equal to the positive externality) conditional on winning. However, then any bidder slightly below the marginal bidder would also like to bid the reserve price, contradicting the definition of the marginal bidder. In our case, this problem does not arise. The gain from dropping the bid to the reserve price for a bidder slightly above the marginal bidder is "small" since it is proportional to the *suprlus* going to a rival whose value is close to that of the marginal bidder. Thus, the marginal bidder will bid his value.

6 Das Varma [4] considers a model in which the externality depends on the rival bidder's identity and shows that the ascending auction generates higher expected revenue than the first-price auction. The literature on externality and auctions is a "reduced form" representation of the effect of potential downstream competition among the participants in an auction. Katzman and Rhodes-Kropf [14] and Goeree [6] consider the signalling incentives of firms bidding for an innovation when they face potential downstream interaction with competitors.

Our analysis is also related to the literature on collusion in auctions (Graham and Marshall [7], McAfee and McMillan [15], Deltas [5]). There is considerable evidence on the prevalence of "bidding rings" and collusive practices in auctions. Members of a bidding rings often organize a private auction among themselves to determine who will get the object if the ring representative wins the object in the public auction, and how the other members of the ring will be compensated. Graham and Marshall [7] and McAfee and McMillan [15] analyze the incentive compatibility and allocative properties of such knockout auctions prior to the public auction. While these papers analyze cartel formation as the instrument of collusion, we analyze a parallel channel, namely, cross-shareholding. Deltas [5] assumes that the winner in the knockout auction is determined after the ring representative has secured the object, and analyses the impact of various surplus-sharing rules on the winning bid in the knockout (which is relevant if the collusion is detected and the seller has to be compensated based on the winning bid in the knockout). To the extent that surplus sharing rules have an impact on the bids, there is a similarity in the issues examined in that paper and ours.

We consider a bidding model in which the bidder's payoff is positive even if he does not participate in the bidding. Such a model has some similarities with models of takeover bidding in which the bidders have toeholds in the target firm. A bidder with toehold may get a positive payoff by not participating, because the other bidder may have to buy out his stake. Burkart [2] and Singh [19] analyze such models in a private value setting, while Bulow, Huang and Klemperer [1] analyze such models in a common value setting. Our analysis enriches theirs by making revenue comparisons in a more general value environment and with general distribution functions. Further, an important aspect of our analysis is to compare the effect of asymmetry on bidding behavior under alternative value environments – an element that is absent in the papers mentioned above.

Finally, there are many papers in the industrial organization and direct foreign investment literature that are concerned with cross-shareholding in the context of standard market games such as Cournot and Bertrand. The aim of these papers is to analyze the impact of cross-shareholding on issues such as technology transfer, innovation, and welfare. Except for noting the collusive consequences of cross-shareholding are preserved when standard market games are replaced by auctions, we do not extend our analysis in this direction. However, some of our results are different from standard market games. For example, in the common value framework, a bidder with more cross-shareholding bids more aggressively relative to the one with less cross-shareholding in second-price auction. This is true neither of Cournot nor Bertrand market games.

3 The general model of cross-shareholding

Suppose there are two firms. We assume that the firms are private firms with each firm i being controlled by an individual shareholder. The controlling shareholder of firm i owns a fraction s_{ij} of the firm j, and a fraction s_{ii} of firm i, with $0 \leq s_{ij} <$

$s_{ii} \le 1$ and $s_{ii} + s_{ji} \le 1$, where $i, j = 1, 2,\ i \ne j$.[7] Note that when $s_{ii} + s_{ji} < 1$, we essentially assume the presence of a third passive shareholder who owns a share $1 - s_{ii} - s_{ji}$ for firm i.[8]

We consider a model in which each firm i can bid for an object after observing privately an informational variable X_i. We assume that X_1, X_2 are independently and identically distributed, i.e. we consider a model of *independent signals.* Let $F(x)$ denote the cumulative distribution of X_i, and let $f(x)$ denote the density, assumed to exist everywhere on the support of $F(x)$. Let $v(x, y)$ denote the value of the object to bidder 1 if $X_1 = x$ and $X_2 = y$, and $v(y, x)$ the corresponding value to bidder 2. We assume that $v(x, y)$ is increasing in both arguments, $v(x, y)$ is non-negative for all realizations of x and y, and

$$\frac{\partial v\left(x,y\right)}{\partial x} \ge \frac{\partial v\left(x,y\right)}{\partial y}. \tag{1}$$

Finally, we define

$$s_1 \equiv \frac{s_{12}}{s_{11}} \text{ and } s_2 \equiv \frac{s_{21}}{s_{22}}$$

which measure the degree of cross-shareholding relative to own-shareholding for bidder 1 and bidder 2, respectively. Note that under our assumptions, $0 \le s_i < 1$, for $i = 1, 2$. In subsequent discussion, we shall refer to a change in the degree of cross-shareholding as simply a change in the cross-shareholding or the cross-ownership of a bidder.

4 Auctions with symmetric cross-shareholdings

Initially, we focus on the symmetric case of $s_{11} = s_{22}$ and $s_{12} = s_{21}$, which implies $s_1 = s_2$ $(= s$, say).[9] Rather than attempting to derive an optimal mechanism for the seller, in the spirit of Milgrom and Weber [16], we shall mainly compare two standard auctions, the first-price (also known as sealed bid) auction and the second-price (which is, under our setting, equivalent to ascending) auction. It is well known that in our setting, if there were no cross-shareholding, these auctions would generate the same expected revenue for the seller for any given reserve price

[7] The main difference between a private firm and public firm specification is that for a private firm, the objective function is fairly unambiguous: the majority owner/manager maximizes his own overall profit, which is the share of profits in his own firm plus that in the rival firm, i.e. the owner of firm i's objective is to maximize : $s_{ii}\times$[Firm i's profit] + $s_{ij}\times$[Firm j's profit]. If firm i were a public firm, a reasonable objective function might be: [Firm i's profit] + $s_{ij}\times$[Firm j's profit]. All our conclusions about the private firm specification remain valid for this particular specification of the public firm's objective.

[8] In an earlier version of our paper, we imposed an adding-up restriction of $s_{ii} + s_{ji} = 1$ so that the existence of such a third shareholder is assumed away. Though similar results can be obtained from either formulation, the current one is more general and provides a more clear understanding of the effects of changes in cross-shareholding and own-shareholding. We are indebted to an anonymous referee for bringing up this point.

[9] Indeed, it suffices to assume $s_1 = s_2$.

set by the seller. As we shall see, this is no longer true when cross shareholdings are present, even though the bidders still may be symmetric. We shall continue to assume also that the seller can set any reserve price at or above zero.

4.1 First-price auction

Without loss of generality, let us look at the problem from bidder $1'$s perspective, where bidder 1 refers to firm 1's controlling shareholder. To derive the equilibrium, consider any monotonically increasing, differentiable bidding strategy $b^F(x)$ in the first-price auction. Let $\Pi(b;z)$ be the expected payoff for bidder 1 who observes $X_1 = z$ and bids b. Then we have:

$$\begin{aligned}\Pi\left(b;z\right) &= s_{11}E\left[\left(v\left(z,X_2\right)-b\right)1_{\{b^F(X_2)<b\}}\right] \\ &\quad +s_{12}E\left[\left(v\left(X_2,z\right)-b^F\left(X_2\right)\right)1_{\{b^F(X_2)>b\}}\right] \\ &= s_{11}\int_{\underline{x}}^{b^{F-1}(b)}\left(v\left(z,\alpha\right)-b\right)f\left(\alpha\right)d\alpha \\ &\quad +s_{12}\int_{b^{F-1}(b)}^{\overline{x}}\left(v\left(\alpha,z\right)-b^F\left(\alpha\right)\right)f\left(\alpha\right)d\alpha.\end{aligned}$$

Differentiating with respect to b:

$$\begin{aligned}\Pi_b\left(b;z\right) &= s_{11}\left[\left(v\left(z,b^{F-1}\left(b\right)\right)-b\right)f\left(b^{F-1}\left(b\right)\right)b^{F-1\prime}\left(b\right)-F\left(b^{F-1}\left(b\right)\right)\right] \\ &\quad -s_{12}\left[\left(v\left(b^{F-1}\left(b\right),z\right)-b^F\left(b^{F-1}\left(b\right)\right)\right)f\left(b^{F-1}\left(b\right)\right)b^{F-1\prime}\left(b\right)\right] \\ &= \frac{1}{b^{F\prime}\left(b^{F-1}\left(b\right)\right)}\Big\{s_{11}\Big[\left(v\left(z,b^{F-1}\left(b\right)\right)-b\right)f\left(b^{F-1}\left(b\right)\right) \\ &\quad -b^{F\prime}\left(b^{F-1}\left(b\right)\right)F\left(b^{F-1}\left(b\right)\right)\Big]-s_{12}\Big[\Big(v\left(b^{F-1}\left(b\right),z\right) \\ &\quad -b^F\left(b^{F-1}\left(b\right)\right)\Big)f\left(b^{F-1}\left(b\right)\right)\Big]\Big\}\end{aligned}$$

If b^F is a best reply for 1, we must have $\Pi_b\left(b^F(x);z\right)=0$ when $z=x$. Substituting $b^F(x)$ for b, we have:

$$s_{11}\left[\left(v(x,x)-b^F(x)\right)f(x)-b^{F\prime}(x)F(x)\right]-s_{12}\left[\left(v(x,x)-b^F(x)\right)f(x)\right]=0.$$

Rearranging terms leads to a first-order linear differential equation:

$$b^{F\prime}(x)=(1-s)\left[v(x,x)-b^F(x)\right]\frac{f(x)}{F(x)}. \tag{2}$$

The solution of this differential equation with the appropriate boundary condition $b^F(x^*)=r$ gives the following expression for the bid function in the first-price auction. Since the differential equation depends on s_{11} and s_{12} only through their ratio, so does the bid function:

Theorem 1. *In the first-price auction with symmetric cross-holdings,*

$$b^F(x) = \frac{rF^{1-s}(x^*) + \int_{x^*}^{x} v(t,t)dF^{1-s}(t)}{F^{1-s}(x)}. \tag{3}$$

Proof. Please see the Appendix.

Note that $1-s \leq 1$ and is decreasing in s, hence $b^F(x)$ is decreasing in cross-shareholding s_{12} and increasing in own-shareholding s_{11}. That is, the higher the degree of cross holding, the less aggressive will be the bidding. In particular, as s_{12} tends to s_{11}, $b^F(x)$ approaches r.

Corollary 2. *In the first-price auction with symmetric cross-shareholding, $b^F(x)$ decreases as s_{12}/s_{11} increases. Hence, the expected profit for the seller decreases as s_{12} increases or s_{11} decreases.*

Proof. Please see the Appendix.

4.2 Second-price auction

In the second-price auction, let $\Pi(b;z)$ denote the expected payoff for bidder 1 who observes $X_1 = z$ and bids b. Then we have:

$$\begin{aligned}\Pi(b;z) &= s_{11}E\left[(v(z,X_2) - b^S(X_2))1_{\{b^S(X_2)<b\}}\right] \\ &\quad + s_{12}E\left[(v(X_2,z) - b)1_{\{b<b^S(X_2)\}}\right] \\ &= s_{11}\int_{\underline{x}}^{b^{S-1}(b)} \left[v(z,\alpha) - b^S(\alpha)\right] f(\alpha)d\alpha + s_{12}\int_{b^{S-1}(b)}^{\overline{x}} \left[v(\alpha,z) - b\right] f(\alpha)d\alpha.\end{aligned}$$

Differentiating with respect to b,

$$\begin{aligned}\Pi_b(b;z) &= s_{11}\frac{db^{S-1}(b)}{db}\left[v\left(z,b^{S-1}(b)\right) - b^S\left(b^{S-1}(b)\right)\right] f\left(b^{S-1}(b)\right) \\ &\quad - s_{12}\int_{b^{S-1}(b)}^{\bar{x}} f(\alpha)d\alpha - s_{12}\frac{db^{S-1}(b)}{db}\left[v\left(b^{S-1}(b),z\right) - b\right] f\left(b^{S-1}(b)\right) \\ &= \frac{s_{11}}{b^{S\prime}\left(b^{S-1}(b)\right)}\left[v\left(z,b^{S-1}(b)\right) - b^S\left(b^{S-1}(b)\right)\right] f\left(b^{S-1}(b)\right) \\ &\quad - s_{12}[1 - F(b^{S-1}(b))] - \frac{s_{12}}{b^{S\prime}(b^{S-1}(b))}[v(b^{S-1}(b),z) - b]f(b^{S-1}(b)).\end{aligned}$$

In equilibrium, $\Pi_b\left(b^S(x);x\right) = 0$ and hence

$$b^{S\prime}(x) = \left(\frac{1}{s} - 1\right)\left[v(x,x) - b^S(x)\right]\frac{f(x)}{[1-F(x)]}. \tag{4}$$

Theorem 3. *In the second-price auction with symmetric cross-holdings,*

$$b^S(x) = \frac{r[1-F(x^*)]^{-\left(\frac{1}{s}-1\right)} + \int_{x^*}^{x} v(t,t)d[1-F(t)]^{-\left(\frac{1}{s}-1\right)}}{[1-F(x)]^{-\left(\frac{1}{s}-1\right)}}. \tag{5}$$

Proof. Please see the Appendix.

Notice once again that as s_{12} tends to s_{11}, $b^S(x)$ tends to r.

4.3 Revenue comparison

An important issue in auction problems is comparison of the expected price received by the seller in alternative auction mechanisms. A well known result (Harris and Raviv [9], Riley and Samuelson [18], Myerson [17]) is that when the signals are independently and identically distributed, the first-price and the second-price auctions yield identical expected revenue for the seller. This is easily verified in our context when $s_{12} = 0$.

To make revenue comparison for $s_{12} > 0$, we have to derive expressions for expected revenue under each auction. A convenient way is to apply the Revelation Principle. Let $\pi^A(z;x)$ denote the expected payoff for the bidder who receives signal x and bids as if he receives signal z. Then,

$$\pi^A(z;x) = s_{11} \int_{\underline{x}}^{z} v(x,\alpha) f(\alpha) d\alpha + s_{12} \int_{z}^{\overline{x}} v(\alpha, x) f(\alpha) d\alpha - P^A(z)$$

where $P^A(z)$ denote the expected payment condition on receiving signal z, for $A = F, S$ denoting, respectively, the first-price and second-price auctions. Differentiating w.r.t. z, we obtain

$$\pi^{A\prime}(z;x) = s_{11} v(x,z) f(z) - s_{12} v(z,x) f(z) - P^{A\prime}(z).$$

In equilibrium, $\pi^{A\prime}(x;x) = 0$, and hence

$$P^{A\prime}(x) = (s_{11} - s_{12}) v(x,x) f(x).$$

Let x^* denote the marginal bidder. Integrating, we get

$$P^A(x) = P^A(x^*) + (s_{11} - s_{12}) \int_{x^*}^{x} v(\alpha,\alpha) f(\alpha) d\alpha \tag{6}$$

for $x > x^*$, where

$$P^F(x^*) = s_{11} r F(x^*) + s_{12} \int_{x^*}^{\overline{x}} b^F(\alpha) f(\alpha) d\alpha$$

and

$$P^S(x^*) = s_{11} r F(x^*) + s_{12} \int_{x^*}^{\overline{x}} r f(\alpha) d\alpha.$$

For $x < x^*$,

$$P^F(x) = s_{12} \int_{x^*}^{\overline{x}} b^F(\alpha) f(\alpha) d\alpha \tag{7}$$

and

$$P^S(x) = s_{12} \int_{x^*}^{\overline{x}} r f(\alpha) d\alpha. \tag{8}$$

Let P^A and R^A be the expected payment and expected revenue respectively, for $A = F, S$. Then, $P^A = \int_{\underline{x}}^{\overline{x}} P^A(x) f(x) dx$. Due to symmetry, in the absence of other passive shareholders, we would simply have $R^A = 2P^A$. In general, the expected payments from both bidders constitute only a fraction of the total expected revenue. In particular, since the expected payment is directly proportional to shares, it is not difficult to see that $R^A = \frac{2}{s_{11}+s_{12}} P^A$. The following result can now be shown:

Theorem 4. *For $s_{12} \in (0, s_{11})$, expected seller revenue in the first-price auction is strictly higher than that in the second-price auction.*

Proof. Please see the Appendix.

As can be seen from the proof, this result is due to the fact that, for any given reserve price, the reservation payoff to the marginal bidder (who is the same for both auctions given the same reserve price) is higher in the second-price auction than in the first-price auction. This is so because by not participating, the marginal bidder can get a share of the other firm's profit, which in the first-price auction is the value of the object to the other bidder, less the price bid by the other bidder (which is greater than the reserve price). On the other hand, in the second-price auction, this latter profit is the difference between the value to the other bidder and the reserve price. Thus, our result is consistent with the observation made in Milgrom and Weber [16] in the context of explaining the revenue equivalence result that for this result to hold, two conditions must be satisfied: (a) the probability of winning for any given realization of the private signals must be the same, and (b) the expected payment made by the marginal bidder must be the same (Milgrom and Weber, [16], page 1092).

We illustrate the breakdown of revenue equivalence with a simple example. Suppose that valuations are private (i.e. $v(x, y) = x$), independent, and uniformly distributed on the unit interval, and that the reserve price is set so that all types participate ($x^* = 0$). In the first price auction, direct substitution gives

$$b^F(x) = \frac{s_{11} - s_{12}}{2s_{11} - s_{12}} x$$

and hence $P(0) = \frac{s_{12}}{2} \frac{s_{11}-s_{12}}{2s_{11}-s_{12}}$. Then, using (6) and simplyfing, we have

$$R^F = \frac{2(1-s)}{3(2-s)}$$

which is decreasing in s. Similarly, we can obain

$$R^S = \frac{1-s}{3(1+s)}$$

which is also decreasing in s. Finally, note that

$$R^F = \frac{2(1+s)}{2-s} R^S > R^S.$$

4.4 Optimal reserve prices: the case of private values model with symmetric cross-shareholding

In the previous sections, we saw that the effect of an increase in the cross-shareholding of all firms is to cause the bidders to bid less aggressively. One might conjecture that this would cause the seller to adjust the reserve price higher in an effort to elicit higher bids. In this section, we confirm that this is indeed the case for both the first and the second-price auctions in the case of independent private values. We also find that the reserve price in the second-price auction is uniformly higher than in the first-price auction.

It is well known that in the independent private values model (i.e. $v(x, y) = x$), the optimal auction requires the seller to impose a nontrivial reserve price. Specifically, the optimal reserve price is implicitly defined as

$$r^* = \frac{1 - F(r^*)}{f(r^*)}.$$

For example, suppose values are uniformly distributed on the unit interval. Then, $r^* = 1/2$.

Denote r^F and r^S respectively the optimal reserve price under the first- and second- price auctions. In the Appendix, we show that with symmetric cross-shareholdings and private values, the reserve price is implicitly given by the following expression in the first-price auction:

$$r^F = \frac{F^{-s}\left(r^F\right) - F\left(r^F\right)}{(1+s)f\left(r^F\right)}. \tag{9}$$

and in the second-price auction, it is given by:

$$r^S = \frac{\left[s + F\left(r^S\right)\right]\left[1 - F\left(r^S\right)\right]}{(1+s)F\left(r^S\right)f\left(r^S\right)}. \tag{10}$$

The following proposition establishes that as the degree of cross-shareholding increases and bidding becomers less aggressive, the seller raises his reserve price in both the first-price and second-price auctions:

Proposition 5. *In both the first-price and second-price auctions with private values, the optimal reserve price is an increasing function of the symmetric cross-shareholding ratio, s, i.e., the optimal reserve price increases as cross-shareholding s_{12} increases and own-shareholding s_{11} decreases.*

Proof. Please see the Appendix.

However, given any degree of cross-shareholding, the choice of optimal reserve price depends on the auction format. We have seen that the expected seller revenue in the first-price auction is strictly higher than that in the second-price auction. In other words, the second-price auction is more vulnerable to the presence of cross-shareholding. Given this, in order to maximize the expected sale price, the seller may be expected to set a higher reserve price in the second-price auction for any given degree of cross-shareholding. The following proposition confirms this intuition.

Proposition 6. *Consider a private values model with symmetric cross-shareholding. Assume that the first and second order necessary conditions for optimum hold at a unique reserve price for both the first and the second-price auction. Then the optimal reserve price under the second-price auction is higher than that under the first-price auction.*

Proof. Please see the Appendix.

To illustrate the result and compare the reserve prices in the two auctions, let us consider an example where values are uniformly distributed on the unit interval ($F(x) = x$, for $x \in [0, 1]$). It is straightforward to show that Equation (9) reduces to:

$$r^F = \left(\frac{1}{2+s}\right)^{\frac{1}{1+s}}$$

which is increasing in s. When $s = 0$ we have $r^F = 0.5$. However, when $s = 0.2$, $r^F = 0.51838$.

From Equation (10), we get:

$$r^S = \frac{1 - s + \sqrt{(1 + 6s + 5s^2)}}{2\,(2 + s)}$$

which is increasing in s. Notice also that $r^S > r^F$.

5 Auctions with asymmetric cross-shareholding

When the cross-shareholdings are asymmetric, we cannot completely characterize the bidding behavior of the firms in the general value environment introduced in Section 3.[10] Nevertheless, we can address the question of which of the firms will bid more aggressively in the auction. We show that for the first-price auction, the bidder with the higher cross-shareholding will bid less aggressively than his rival. For the second-price auction, however, this is not necessarily the case. In fact, which bidder bids more aggressively depends on the value environment. In this section, we shall show that when we specialize our model to the independent private values model, the bidder with the higher cross-shareholding does indeed bid less aggressively in the second-price auction for sufficiently low realizations of the signal. Moreover, again for the private values model, if one of the bidders has zero cross-shareholding, we are able to show that the bidder with positive cross-shareholding bids less aggressively. However, in the next section, we specialize the model to the case of the pure common value model (with independent private signals) and show the counter-intuitive result that the bidder with the higher cross-shareholding bids *more* aggressively in the second-price auction.

[10] However, later, we shall specialize the value environment and explicitly solve for the bid functions.

5.1 First-price auction

In this section, we assume, for simplicity, $r = v(\underline{x}, \underline{x})$. In the first-price auction, the expected payoff for bidder i is given by

$$\Pi_i^F(b; z) = s_{ii} \int_{\underline{x}}^{b_j^{F-1}(b)} (v(z, \alpha) - b) f(\alpha) d\alpha + s_{ij} \int_{b_j^{F-1}(b)}^{\overline{x}} \left(v(\alpha, z) - b_j^F(\alpha)\right) f(\alpha) d\alpha.$$

By standard arguments, a necessary condition for an equilibrium is that the boundary conditions are $b_i^F(\underline{x}) = b_j^F(\underline{x}) = v(\underline{x}, \underline{x})$ and $b_i^F(\overline{x}) = b_j^F(\overline{x})$. Differentiating with respect to b gives the following first-order condition

$$b_j^{F\prime}\left(\phi_j^F(x_i)\right) = \left[\frac{s_{ii}\left[v\left(x_i, \phi_j^F(x_i)\right) - b_i^F(x_i)\right] - s_{ij}\left[v\left(\phi_j^F(x_i), x_i\right) - b_i^F(x_i)\right]}{s_{ii}}\right] \times \frac{f\left(\left(\phi_j^F(x_i)\right)\right.}{F\left(\left(\phi_j^F(x_i)\right)\right.}$$

where $\phi_j^F(x_i) = b_j^{F-1}\left(b_i^F(x_i)\right)$. The differential equation for the other bidder can be derived symmetrically. This pair of differential equations constitutes another necessary condition for an equilibrium. In equilibrium, we must have $x_j = \phi_j^F(x_i)$ and $x_i = \phi_i^F(x_j)$, and hence we obtain the following system of differential equations:

$$b_j^{F\prime}\left(\phi_j^F(x_i)\right) = \left[\frac{s_{ii}\left[v\left(x_i, \phi_j^F(x_i)\right) - b_i^F(x_i)\right] - s_{ij}\left[v\left(\phi_j^F(x_i), x_i\right) - b_i^F(x_i)\right]}{s_{ii}}\right] \times \frac{f\left(\left(\phi_j^F(x_i)\right)\right.}{F\left(\left(\phi_j^F(x_i)\right)\right.} \tag{11}$$

and

$$b_i^{F\prime}(x_i) = \left[\frac{s_{jj}\left[v\left(\phi_j^F(x_i), x_i\right) - b_j^F(\phi_j(x_i))\right] - s_{ji}\left[v\left(x_i, \phi_j^F(x_i)\right) - b_j^F\left(\phi_j^F(x_i)\right)\right]}{s_{jj}}\right] \times \frac{f(x_i)}{F(x_i)}. \tag{12}$$

Without further specification, one cannot obtain explicit solutions. However, we are able to prove the following. The proof is contained in the Appendix.

Proposition 7. *In the first-price auction, the bidder with less cross-holding will bid more aggressively. In particular,*

$$s_i < s_j$$

implies

$$b_i^F(x) > b_j^F(x).$$

An increase in s_{ij} or a decrease in s_{ii} makes bidder i less keen to bid high, since losing is less costly. However, whether or not, *in equilibrium*, bidder i will bid less aggressively than bidder j, or whether bidder i will bid lower following an increase in his degree of cross-shareholding, is a considerably more complicated issue. Proposition 7 shows that, for the first-price auction, bidder i does bid less aggressively than bidder j when his degree of cross-shareholding is higher. Our subsequent discussion will show that this need not be the case for the second-price auction.

5.2 Second-price auction

In the second-price auction, we have:

$$\Pi_i^S(b;z) = s_{ii}\int_{\underline{x}}^{b_j^{S-1}(b)} \left[v(z,\alpha) - b_i^S(\alpha)\right] f(\alpha)d\alpha$$
$$+s_{ij}\int_{b_j^{S-1}(b)}^{\overline{x}} [v(\alpha,z) - b]f(\alpha)d\alpha.$$

The appropriate boundary conditions are $b_i^S(\underline{x}) = b_j^S(\underline{x}) = v(\underline{x},\underline{x})$ and $b_i^S(\overline{x}) = b_j^S(\overline{x})$. Moreover, the equilibrium bid functions satisfy the following system of differential equations:

$$b_j^{S\prime}(\phi_j^S(x_i)) \tag{13}$$
$$= \left[\frac{s_{ii}\left[v\left(x_i,\phi_j^S(x_i)\right) - b_i^S(x_i)\right] - s_{ij}\left[v\left(\phi_j^S(x_i),x_i\right) - b_i^S(x_i)\right]}{s_{ij}}\right]$$
$$\times\frac{f\left(\phi_j^S(x_i)\right)}{1-F\left(\phi_j^S(x_i)\right)}$$

and

$$b_i^{S\prime}(x_i) \tag{14}$$
$$= \left[\frac{s_{jj}\left[v\left(\phi_j^S(x_i),x_i\right) - b_j^S\left(\phi_j^S(x_i)\right)\right] - s_{ji}\left[v\left(x_i,\phi_j^S(x_i)\right) - b_j^S\left(\phi_j^S(x_i)\right)\right]}{s_{ji}}\right]$$
$$\times\frac{f(x_i)}{1-F(x_i)}.$$

where $\phi_j^S(x_i) = b_j^{S-1}\left(b_i^S(x_i)\right)$.

In the case of second-price auction, as we shall see below, it will turn out that which bidder bids more aggressively depends on the value environment. Here, we show that in a special case of the private value model, the bidder with less cross-holding will bid more aggressively. We shall see in the next section that for the common values model (a special case of which is subsumed in our general value environment), we have the opposite result, and the bidder with less cross-holding bids less aggressively.

Proposition 8. *Consider the private values model and set one bidder's cross-holding to zero (say, $s_{ij} = 0$ and $s_{ji} > 0$). Moreover, assume that the private values are distributed uniformly over the unit interval (i.e. $F(x) = x$). Then in the second-price auction, the bidder with positive cross-holding bids less aggressively, i.e. $b_j(x) < b_i(x)$.*

Proof. Please see the Appendix.

Proposition 8 imposes restrictions on the cross-holding of one of the individuals, as well as the distribution of the private values. Without imposing these restrictions, one can show that at least for low realizations of the private value, the bidder with the higher cross-holding will bid less aggressively.

Proposition 9. *For the private values model, the bidder with higher cross-holding will bid less aggressively in the second-price auction for sufficiently low realizations of the private value. In particular,*

$$s_i < s_j$$

implies

$$b_i(x) > b_j(x)$$

for x sufficiently close to $\underline{x}$.

Proof. Please see the Appendix.

Proposition 9 implies that in the private value model, at least for some realizations of the private signals, the bidder with more cross-shareholding will bid less aggressively in the second-price auction. Proposition 8 shows that for a special case of asymmetric cross-shareholding, the bidder with less cross-shareholding does bid more aggressively for *all* realizations of the private signals. Thus, it will never be the case that, in the private values environment, the bidder with more cross-shareholding will bid more aggressively. However, as we shall see in the next section, in the common value environment, where the winner's curse plays an important role, this result for the second-price auction is reversed.

6 The pure common value model

In this section, we extend our analysis to the "pure common value" model explored by Bulow et al. [1] to further examine the role of asymmetries in cross-shareholding. In this common value model, we have $v(x, y) = v(y, x)$, i.e. the value of the object to either bidder depends only on the private signals, and not on who draws which signal. This model is not contained in the general model we have developed; however, a special case of this model is subsumed in the general model.[11]

The advantage of considering this particular model is that we are able to obtain closed form solutions for most variables of interest. This enables us to explicitly investigate the properties of the bidding strategies as functions of the cross-shareholdings as well as make revenue comparisons when the cross-sharholdings are not identical – something we could not do for the general model. The main general conclusions that emerge from this analysis are the following. In the presence of asymmetries in cross-shareholding: (i) bidding behavior in the first-price auction is more robust to changes in the value environment than in the second-price auction (ii) in the common value setting, bidding outcomes in the second-price auction are much more sensitive to changes in cross-shareholding than the first-price auction, (iii) in the common value setting, expected seller revenue increases as cross-shareholding becomes more asymmetric in the first-price auction, whereas exactly the opposite is the case with the second-price auction, and (iv) in the common value setting, irrespective of the nature of cross-shareholding, expected seller revenues are higher in the first-price auction than in the second-price auction.

To obtain close-form solutions, we assume that $F(x)$ is uniform over the unit interval. For the first-price auction, we can proceed as we did in deriving Equations (11) and (12). Using the definition $\phi_j(x_i) = b_j^{-1}(b_i(x_i))$ and $v(x, y) = v(y, x)$, we get:

$$b'_j(\phi_j(x_i)) = (1 - s_i)\left[v(\phi_j(x_i), x_i) - b_i(x_i)\right] \frac{1}{\phi_j(x_i)} \tag{15}$$

and

$$b'_i(x_i) = (1 - s_j)\left[v(\phi_j(x_i), x_i)) - b_j(\phi_j(x_i))\right] \frac{1}{x_i}. \tag{16}$$

Unlike the general model, here we are able to obtain close-form solutions for the bid functions even when the cross-holdings are not equal:

[11] The pure common value model proposed here is not a proper subset of the "general model". For example, in our general symmetric model, we may have

$$v_1(x_1, x_2) = \alpha x_1 + (1 - \alpha)x_2 \text{ and } v_2(x_2, x_1) = \alpha x_2 + (1 - \alpha)x_1.$$

i.e. both bidders put a weight α on his signal. But in the pure common values model considered in this section, we allow

$$v_1(x_1, x_2) = \alpha x_1 + (1 - \alpha)x_2 \text{ and } v_2(x_2, x_1) = \alpha x_1 + (1 - \alpha)x_2.$$

i.e. both bidders put more weight on bidder 1's signal. The intersection of the two model is when $\alpha = 1/2$. More generally, the intersection is when the function $v_i(x, y)$ is symmetric about x and y.)

Proposition 10. *For the common value model with uniformly distributed signal, the bid function for bidder i in the first-price auction is given by*

$$b_i(x_i) = (1 - s_j)\frac{\int_0^{x_i} v(t, \phi_j(t))t^{-s_j}\,dt}{{x_i}^{1-s_j}}$$

where

$$\phi_j(t) = t^{\frac{1-s_j}{1-s_i}}.$$

Moreover, $s_i < s_j \Rightarrow b_i(x) > b_j(x)$, *i.e. the bidder with the lower cross-holding will bid more aggressively.*

Proof. Please see the Appendix.

Proposition 10 confirms the result of Proposition 7 that the bidder with higher cross-shareholding bids less aggressively in the first-price auction.[12] Moreover, whereas earlier we could not explicitly solve for the bid function and therefore could not say whether an increase in cross-shareholding causes a bidder to bid less aggressively than before, here, in the common value setting, we are able to do so. Note that $v(t, t^{\frac{1-s_j}{1-s_i}})$ and hence $b_i(x_i)$ is decreasing in s_i. Therefore, $b_i(x_i)$ is decreasing in s_{ij} and increasing in s_{ii}. However, since both $v(t, t^{\frac{1-s_j}{1-s_i}})$ and $(\frac{t}{x_i})^{1-s_j}$ are increasing in s_j, it can not be shown in general how $b_i(x_i)$ changes as s_j changes.

While we cannot in general say whether any bidder's bid is monotonic in his rival's cross-holding, notice that bidder i wins with probability

$$\int_0^1 \int_0^{x^{\frac{1-s_j}{1-s_i}}} dt dx = \frac{(1 - s_i)}{(1 - s_i) + (1 - s_j)}$$

which is decreasing in s_i and increasing s_j. In other words, the ex-ante probability for bidder i to win is decreasing in his share of the other firm s_{ij} and the other bidder's own share of his firm s_{jj}, but increasing in his own share s_{ii} and the other bidder's share of his firm s_{ji}.[13]

Note also that if $s_i < s_j$,

$$\frac{(1 - s_i)}{(1 - s_i) + (1 - s_j)} - \frac{(1 - s_j)}{(1 - s_i) + (1 - s_j)} = \frac{s_j - s_i}{(1 - s_i) + (1 - s_j)} > 0$$

i.e. the bidder with smaller cross-shareholding is more likely to win in the first-price auction, which, of course, must be true since he bids more aggressively.

[12] However, notice that the common value setting is not completely subsumed in the general value environment of Proposition 7 – only a particular case of the common value model is (see Footnote 11). Thus, the result of Proposition 10 is not implied by Proposition 7.

[13] Notice that when the cross-holdings are symmetric, both bidders must win equally often. Thus, the bidder will bid higher than his rival more than 50% of the time if his cross-shareholding is lower than that of the rival, and the probability that he bids higher than the rival is also decreasing in his cross-shareholding.

Under the second-price auction, proceeding as in the derivation of first-order conditions (13) and (14), we get:

$$b_i'(x_i) = \frac{1-s_j}{s_j}\frac{1}{1-x_i}\left[v(\phi_j(x_i),x_i) - b_j(\phi_j(x_i))\right]$$

and

$$b_j'(\phi_j(x_i)) = \frac{1-s_i}{s_i}\frac{1}{1-\phi_j(x_i)}\left[v(\phi_j(x_i),x_i) - b_i(x_i)\right]$$

with $s_i, s_j > 0$.

Solving the pair of differential equations gives us the following:

Proposition 11. *For the common value model with uniformly distributed signal, the bid function for bidder i in the second-price auction is given by*

$$b_i(x_i) = \frac{1-s_j}{s_j}\frac{\int_0^{x_i} v(t,\phi_j(t))(1-t)^{-\frac{1}{s_j}}\,dt}{(1-x_i)^{-\frac{1-s_j}{s_j}}}$$

where

$$\phi_j(t) = 1-(1-t)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}}.$$

Moreover, $s_i < s_j \Rightarrow b_i(x) < b_j(x)$, *i.e. the bidder with the lower cross-holding will bid less aggressively.*

Proof. Please see the Appendix.

Note that $\phi_j(t) = 1-(1-t)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}}$ is increasing in s_i and hence so is $b_i(x)$. In other words, $b_i(x)$ is increasing in s_{ij} and decreasing in s_{ii}.

Moreover, the probability with which bidder i wins is increasing in s_i (but decreasing in s_j):

$$\int_0^1\int_0^{1-(1-x_i)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}}} dt dx_i = \frac{s_i(1-s_j)}{s_i(1-s_j)+s_j(1-s_i)}$$

which is increasing in s_i and decreasing in s_j. Thus, the ex-ante probability of winning for bidder i is increasing in s_{ij} and s_{jj}, but decreasing in s_{ii} and s_{ji}.

For $s_i < s_j$, we also have:

$$\begin{aligned}&\frac{s_i(1-s_j)}{s_i(1-s_j)+s_j(1-s_i)} - \frac{s_j(1-s_i)}{s_i(1-s_j)+s_j(1-s_i)}\\ &= \frac{s_i - s_j}{s_i(1-s_j)+s_j(1-s_i)} < 0\end{aligned}$$

i.e., if bidder i has the smaller cross-holding, he is less likely to win than bidder j in the second-price auction, which must be true as he bids less aggressively.

Proposition 11 indicates that, in stark contrast to the first price auction, for the second price auction, the bidder with higher cross-holding is more aggressive. Notice, however, that for the private values model, we know from Propositions 8 and 9 that such a result cannot hold. Thus, who bids more aggressively in the second price auction is sensitive to the value environment.

The reason why for the second-price auction we get the counterintuitive result that the bidder with more cross shareholding bids more agressively has to do winner's curse phenomenon that is typical of the common value environment. To understand why the results for the first-price and the second-price auctions differ, suppose firm i's cross-shareholding increases. In both auctions, he has an incentive to bid lower. The fact that firm i bids low implies that, conditional on winning, the object must be more valuable to j, ceteris paribus. This should cause j to bid higher to win. This incentive may be greater in the second-price auction, since bidding high does not cost as much conditional on winning as in the first-price auction (in the second-price auction, the bidder pays the other bidder's bid conditional on winning).

However, if j bids very high, i now suffers from winner's curse. This would cause i to bid even lower, further reinforcing j's incentive to bid higher. Thus, it is possible that an equilibrium may not exist in the second-price auction in which i bids lower than j.

On the other hand, an equilibrium may exist in which j bids lower than i in the second-price auction. The equilibrium is sustained because in equilibrium, bidder i has the stronger incentive to bid high. The reason is that it is more attractive for i to win, given than j bids low, since the object will be worth more conditional on winning. Such an equilibrium may be more difficult to sustain in a first-price auction, since i may want to trade-off the higher probability of winning by lowering his bid and reducing the cost of winning. Thus, we may have equilibria in which i bids less than j only in the first-price auction.

Finally, to note one further implication of our analysis, suppose $s_i < s_j$, and let us flip the auction rule from first to second-price auction. Under the first-price auction, bidder i bids more aggresively, and has the higher probability of winning. Under the second-price auction, bidder j bids more aggressively and has the higher probability of winning. The difference in the probabilities of winning between bidder i in the first-price auction and bidder j in the second- price auction is:

$$\begin{aligned}&\frac{(1-s_i)}{(1-s_i)+(1-s_j)} - \frac{s_j(1-s_i)}{s_i(1-s_j)+s_j(1-s_i)}\\ &= (1-s_i)\,(1-s_j)\,\frac{s_i - s_j}{(2-s_i-s_j)\,(s_i - 2s_is_j + s_j)} < 0.\end{aligned}$$

Since the ex-ante probability of winning is equal (0.5) if $s_i = s_j$, this shows that the probability of winning of the bidder who is more likely to win is more sensitive to changes in cross-holdings in the second-price auction than in the first-price auction. In the common value model, bidding outcomes are more sensitive to changes in cross shareholding for the second-price than the first-price auction.

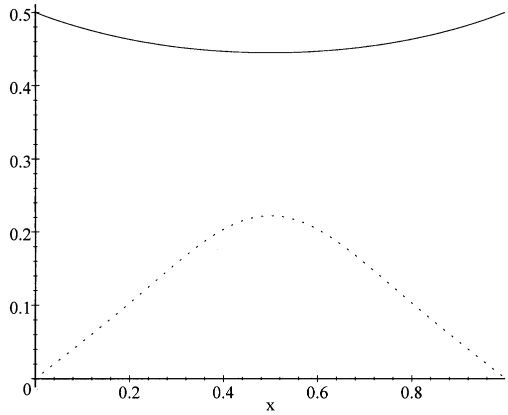

Figure 1. Solid curve represents expected revenue in first-price auction, dashed curve represents expected revenue in second-price auction

6.1 Expected revenue comparison

To compare the expected revenue to the seller under the two auctions, we further specialize the model by assuming that $v(x_i, x_j) = x_i + x_j$. With this assumption, it can be shown that (details are given in the Appendix)

$$P^F(s_1, s_2) = \frac{(2 - s_1 - s_2)\left(12 + 5s_1^2 + 5s_2^2 + 14s_1s_2 - 15s_1 - 15s_2 - 3s_1^2s_2 - 3s_1s_2^2\right)}{(2 - s_1)(2 - s_2)(3 - s_1 - 2s_2)(3 - 2s_1 - s_2)}$$

and

$$P^S(s_1, s_2) = \frac{(1 - s_1)(1 - s_2)\left(s_1^2 + s_2^2 + 6s_1^2s_2^2 + 4s_1s_2 - 6s_1^2s_2 - 6s_1s_2^2\right)}{(1 - s_1s_2)(s_1 + 2s_2 - 3s_1s_2)(2s_1 + s_2 - 3s_1s_2)}.$$

Let $s_1 + s_2 = \overline{s}$. Upon substitution, it is easy to see that $P^F(s_1, \overline{s} - s_1)$ is minimized at $s_1 = \overline{s}/2$; whereas $P^S(s_1, \overline{s} - s_1)$ is maximized at $s_1 = \overline{s}/2$. Moreover, $P^F(s_1, \overline{s} - s_1)$ is decreasing in s_1 for $s_1 < \overline{s}/2$ and increasing for $s_1 > \overline{s}/2$; and $P^S(s_1, \overline{s} - s_1)$ is increasing for $s_1 > \overline{s}/2$ and decreasing for $s_1 > \overline{s}/2$. For example, when $\overline{s} = 1$,we have the situation depicted in Figure 1.

Therefore, we have:

Proposition 12. *For the additive common value model with uniformly distributed signal, increased asymmetry in cross-holding increases expected revenue in the first-price auction, but decreases expected revenue in the second-price auction.*

Since we have shown that under symmetric cross-holding, the first-price auction generates higher expected revenue for the seller, it follows that:

Proposition 13. *For the additive common value model with uniformly distributed signal, for any s_i and s_j, expected seller revenue in the first-price auction is strictly higher than that in the second-price auction.*

7 Conclusion

We analyze bidding behavior by firms when the rival bidders hold shares in each other. With symmetric cross-shareholding, the first-price auction generates higher expected revenue to the seller than the second-price auction, irrespective of whether the value environment is private or common value, and irrespective of the distributional assumptions (as long as the private signals are i.i.d.). This result is different from Bulow, Huang and Klemperer [1] who, in their model of takeover bidding with toeholds in the common value setting, find that when the toeholds are symmetric, the second-price auction generates higher expected revenue. Bulow, Huang and Klemperer [1] find, however, that as the toeholds become more asymmetric, the first-price auction does significantly better – a result that we also find in the common value model. What our result does suggest is that while the winner's curse problem may be important in understanding in which type of auction the expected sale price is more sensitive to a change in toehold or cross-shareholding (to the extent it causes firms to bid more or less aggressively), it (alone) does not explain which auction will ultimately perform better. Put differently, the winner's curse reflects the slope of the curves in Figure 1, but it does not necessarily tell us anything about the levels.

Our results also point to the fact that the first-price auction is more robust to changes in the value environment and to asymmetry than the second-price auction. The bidder with higher cross-shareholding bids less aggressively in the first-price auction irrespective of whether the model is private or common value; in contrast, for the second-price auction, which bidder bids more aggressively seems to depend on the value environment. Finally, in the common value model, bids and the probability of winning are more sensitive to changes in cross-shareholding in the second-price auction than in the first-price auction.

Appendix

Proof of Theorem 1

First, we establish that the appropriate boundary condition is $b^F(x^*) = r$. Since the marginal bidder, with signal x^*, is indifferent between bidding $b^F(x^*)$ and not bidding, we have:

$$s_{11}\int_{\underline{x}}^{x^*}\left[v(x^*,\alpha)-b^F(x^*)\right]f(\alpha)d\alpha+s_{12}\int_{x^*}^{\overline{x}}\left[v(\alpha,x^*)-b^F(\alpha)\right]f(\alpha)d\alpha$$
$$=s_{12}\int_{x^*}^{\overline{x}}\left[v(\alpha,x^*)-b^F(\alpha)\right]f(\alpha)d\alpha$$

or simply

$$\int_{\underline{x}}^{x^*}\left[v(x^*,\alpha)-b^F(x^*)\right]f(\alpha)d\alpha=0. \tag{17}$$

On the other hand, by assumption, $b^F(x^*) \geq r$. Suppose $b^F(x^*) > r$, then

$$\Pi\left(x^*, b^F(x^*)\right) = s_{12} \int_{x^*}^{\overline{x}} \left[v(\alpha, x^*) - b^F(\alpha)\right] f(\alpha) d\alpha.$$

If he bids r instead, he can get

$$\begin{aligned}\Pi(x^*, r) = s_{11} &\int_{\underline{x}}^{x^*} \left[v(x^*, \alpha) - r\right] f(\alpha) d\alpha \\ &+ s_{12} \int_{x^*}^{\overline{x}} \left[v(\alpha, x^*) - b^F(\alpha)\right] f(\alpha) d\alpha \\ &> s_{12} \int_{x^*}^{\overline{x}} \left[v(\alpha, x^*) - b^F(\alpha)\right] f(\alpha) d\alpha\end{aligned}$$

since $\int_{\underline{x}}^{x^*} \left[v(x^*, \alpha) - r\right] f(\alpha) d\alpha > 0$ if $b^F(x^*) > r$, by virtue of equation (17). Hence, he will deviate and therefore, we must have $b^F(x^*) = r$. Notice that this also implies

$$\int_{\underline{x}}^{x^*} \left[v(x^*, \alpha) - r\right] f(\alpha) d\alpha = 0. \tag{18}$$

Next, we need to verify that the first order necessary condition (2) defines a best, rather than a worst decision for the bidder. Consider

$$\begin{aligned}&\Pi_b\left(b^F(x); z\right) \\ &= \frac{s_{11}}{b^{F\prime}(x)} \left[\left(v(z, x) - b^F(x)\right) f(x) - b^{F\prime}(x) F(x)\right] \\ &\quad - \frac{s_{12}}{(b^{F\prime}(x)} \left[\left(v(x, z) - b^F(x)\right) f(x)\right] \\ &= \frac{f(x)}{b^{F\prime}(x)} \left[s_{11} v(z, x) - s_{12} v(x, z) - (s_{11} - s_{12}) b^F(x) - (1 - s) b^{F\prime}(x) \frac{F(x)}{f(x)}\right]\end{aligned}$$

The bracketed expression is zero when $x = z$. Since $s_{11} > s_{12}$, by (1), the bracketed expression and hence $\Pi_b(b^F(x); z)$ has the same sign as $(z - x)$.

Now, using the boundary condition ($b^F(x^*) = r$) we can solve the differential equation (2) for the expression given in the Theorem.

Proof of Corollary 2: (The bid function under the first-price auction is decreasing in the degree of cross-shareholding.)

The bid function can be rewritten as

$$b^F(x) = \int_{\underline{x}}^{x} u(t, t) d\left[\frac{F(t)}{F(x)}\right]^{1-s}$$

where

$$u(t,t) = \begin{cases} r & \text{if } t \leq x^* \\ v(t,t) & \text{if } t > x^* \end{cases}.$$

The result follows from the facts that (i) $[\frac{F(t)}{F(x)}]^{1-s}$ first-order stochastic dominates $[\frac{F(t)}{F(x)}]^{1-s'}$ for $s < s'$, and (ii) using (18), it is easy to check that $v(x^*, x^*) \geq r$, so that $u(t,t)$ is an increasing function.

Proof of Theorem 3

The marginal bidder's expected payoff from bidding $b^S(x^*)$ is

$$s_{11} \int_{\underline{x}}^{x^*} [v(x^*, \alpha) - r] f(\alpha) d\alpha + s_{12} \int_{x^*}^{\overline{x}} \left[v(\alpha, x^*) - b^S(x^*)\right] f(\alpha) d\alpha$$

which is decreasing in $b^S(x^*)$, and since $b^S(x^*) \geq r$, it follows that he must bid r.

Note also that he is indifferent between bidding and not bidding, and so we have

$$\begin{aligned} & s_{11} \int_{\underline{x}}^{x^*} [v(x^*, \alpha) - r] f(\alpha) d\alpha + s_{12} \int_{x^*}^{\overline{x}} [v(\alpha, x^*) - r] f(\alpha) d\alpha \\ & = s_{12} \int_{x^*}^{\overline{x}} [v(\alpha, x^*) - r] f(\alpha) d\alpha, \end{aligned} \tag{19}$$

or simply

$$\int_{\underline{x}}^{x^*} [v(x^*, \alpha) - r] f(\alpha) d\alpha = 0. \tag{20}$$

To see whether the FOC defines a maximum, consider

$$\begin{aligned} & \Pi_b \left(b^S(x); z\right) \\ & = \frac{s_{11}}{b^{S\prime}(x)} \left[v(z,x) - b^S(x))\right] f(x) \\ & \quad -s_{12}[1 - F(x)] - \frac{s_{12}}{b^{S\prime}(x)} \left[v(x,z) - b^S(x)\right] f(x) \\ & = \frac{f(x)}{b^{S\prime}(x)} \\ & \left[[s_{11} v(z,x) - s_{12} v(x,z)] - (s_{11} - s_{12}) b^S(x) - s b^{S\prime}(x) \frac{[1 - F(x)]}{f(x)} \right]. \end{aligned}$$

Once again, by virtue of $s_{11} > s_{12}$ and condition (1), we see that the bracketed expression has the same sign as $z - x$. Thus, from (4), together with the boundary condition $b^S(x^*) = r$, we have the expression for the bid function given in the statement of the Theorem.

Proof of Theorem 4.

Notice that

$$P^F(x^*) = s_{11} r F(x^*) + s_{12} \int_{x^*}^{\overline{x}} b^F(\alpha) f(\alpha) d\alpha \tag{21}$$

and

$$P^S(x^*) = s_{11} r F(x^*) + s_{12} \int_{x^*}^{\overline{x}} r f(\alpha) d\alpha. \tag{22}$$

From Equations (18) and (20), it follows that for any given reserve price r, the corresponding x^* must be the same under both auctions. From equations (6), (7), (8), (21) and (22), it is clear that in the absence of cross-shareholding, $P^F(x) = P^S(x) = 0$ for $x < x^*$ and $P^F(x) = P^S(x) = rF(x^*) + \int_{x^*}^{x} v(\alpha,\alpha) f(\alpha) d\alpha$ for $x \geq x^*$. The revenue equivalence result is therefore established. For $s_{12} > 0$, it suffices to show that $P^F(x) > P^S(x)$ for all x. When $x \leq x^*$, $P^F(x) > P^S(x)$, since $b^F(\alpha) > r$ for $\alpha \in (x^*, \overline{x}]$. For $x \geq x^*$, $P^F(x) \gtreqless P^S(x)$ according as $P^F(x^*) \gtreqless P^S(x^*)$. Since $P^F(x^*) > P^S(x^*)$, the result follows.

Derivation of optimal reserve price under the private value model

We will first obtain an expression for the expected revenue. Using the expressions for P^F and P^A in Section 4.3 and noting that $x^* = r$, we have, under the private values model,

$$\begin{aligned} P^F = {} & s_{12} \int_r^{\overline{x}} b^F(\alpha) f(\alpha) d\alpha + s_{11} r [1 - F(r)] F(r) \\ & + (s_{11} - s_{12}) \int_r^{\overline{x}} \left[\int_r^x \alpha f(\alpha) d\alpha \right] f(x) dx \end{aligned}$$

and

$$\begin{aligned} P^S = {} & s_{12} \int_r^{\overline{x}} r f(\alpha) d\alpha + s_{11} r F(r) [1 - F(r)] \\ & + (s_{11} - s_{12}) \int_r^{\overline{x}} \left[\int_r^x \alpha f(\alpha) d\alpha \right] f(x) dx. \end{aligned}$$

The expected revenue is simply $R^A = \frac{2}{s_{11}+s_{12}} P^A$, $A = F, S$. Under the first-price auction, substituting $b^F(\alpha)$, we have

$$\begin{aligned} & \frac{s_{11}+s_{12}}{2} R^F \\ & = s_{12} \int_r^{\overline{x}} \frac{r F^{1-s}(r) + (1-s) \int_r^{\alpha} \theta F^{-s}(\theta) f(\theta) d\theta}{F^{1-s}(\alpha)} f(\alpha) d\alpha \\ & \quad + s_{11} r F(r) [1 - F(r)] + (s_{11} - s_{12}) \int_r^{\overline{x}} \left[\int_r^x \alpha f(\alpha) d\alpha \right] dF(x) \end{aligned}$$

and hence

$$\frac{1}{2}R^F = \frac{s}{1+s}\int_r^{\overline{x}} \frac{rF^{1-s}(r) + (1-s)\int_r^{\alpha}\theta F^{-s}(\theta)f(\theta)d\theta}{F^{1-s}(\alpha)} f(\alpha)d\alpha$$
$$+\frac{1}{1+s}rF(r)[1-F(r)] + \frac{1-s}{1+s}\int_r^{\overline{x}}\left[\int_r^x \alpha f(\alpha)d\alpha\right]dF(x).$$

Differentiating w.r.t. r and simplfying yields:

$$\frac{1}{2}R^{F\prime} = \frac{1}{1+s}F(r)[F^{-s}(r) - F(r)] - rf(r)F(r).$$

Equating the above expression to zero, we have r^F satisfying the following

$$r^F = \frac{F^{-s}(r^F) - F(r^F)}{(1+s)f(r^F)}.$$

Similarly, we have

$$\frac{1}{2}R^S = \frac{s}{1+s}r[1-F(r)]$$
$$+\frac{1}{1+s}rF(r)[1-F(r)] + \frac{1-s}{1+s}\int_r^{\overline{x}}\left[\int_r^x \alpha f(\alpha)d\alpha\right]dF(x).$$

Differentiating w.r.t. r yields:

$$\frac{1}{2}R^{S\prime} = \frac{1}{1+s}\left[s + (1-s)F(r) - F^2(r)\right] - rf(r)F(r)$$

and hence

$$r^S = \frac{\left[s + F\left(r^S\right)\right]\left[1 - F\left(r^S\right)\right]}{(1+s)F\left(r^S\right)f\left(r^S\right)}.$$

Proof of Proposition 5:
(The optimal reserve price is increasing in the degree of cross-shareholding)

Recall that

$$\frac{\partial R^F(r;s)}{\partial r} = \frac{2}{1+s}F(r)[F^{-s}(r) - F(r)] - rf(r)F(r)$$

It suffices to show that the above expression is increasing in s. Consider

$$\frac{\partial^2 R^F(r;s)}{\partial r\partial s} = 2F(r)\left[-\frac{1}{1+s}F^{-s}(r)\ln F(r) - [F^{-s}(r) - F(r)]\frac{1}{(1+s)^2}\right]$$
$$= \frac{2F^{1-s}(r)}{(1+s)^2}\left[F^{1+s}(r) - \ln F^{1+s}(r) - 1\right]$$

which is positive, since $F^{1+s}(r) - \ln F^{1+s}(r) > 1$ for any distribution function F and r. Thus, the result follows.

Similarly,

$$\frac{\partial^2 R^S(r,s)}{\partial r \partial s} = \frac{2}{1+s}[1-F(r)] - \frac{2}{(1+s)^2}\left[s+(1-s)F(r)-F^2(r)\right]$$
$$= \frac{2[(1-F(r)]^2}{(1+s)^2} > 0.$$

Proof of Proposition 6:
(The optimal reserve price in the second-price auction is higher than that in the first-price auction)

Observe that

$$\frac{1}{2}(R^{F\prime} - R^{S\prime}) = \frac{s}{1+s}\int_r^{\bar{x}} \left[\left(\frac{F(r)}{F(\alpha)}\right)^{1-s} - 1\right] f(\alpha)d\alpha$$

Moreover,

$$\left[\frac{F(r)}{F(\alpha)}\right]^{1-s} < 1$$

for all $\alpha \in (r, \overline{x})$. Therefore, $R^{F\prime} < R^{S\prime}$, and hence $r^F < r^S$, assuming r^F and r^S are unique.

Proof of Proposition 7:
(The bidder with more cross-holding will bid less aggressively in the first-price auction)

Suppose $s_i < s_j$. Recall

$$b_i^{F\prime}(x_i) \tag{23}$$
$$= \frac{s_{jj}\left[v\left(\phi_j^F(x_i), x_i\right) - b_j^F\left(\phi_j^F(x_i)\right)\right] - s_{ji}\left[v\left(x_i, \phi_j^F(x_i)\right) - b_j^F\left(\phi_j^F(x_i)\right)\right]}{s_{jj}}$$
$$\times \frac{f(x_i)}{F(x_i)}$$

and

$$b_j^{F\prime}\left(\phi_j^F(x_i)\right) \tag{24}$$
$$= \frac{s_{ii}\left[v\left(x_i, \phi_j^F(x_i)\right) - b_i^F(x_i)\right] - s_{ij}\left[v\left(\phi_j^F(x_i), x_i\right) - b_i^F(x_i)\right]}{s_{ii}}$$
$$\times \frac{f\left(\left(\phi_j^F(x_i)\right)\right)}{F\left(\left(\phi_j^F(x_i)\right)\right)}$$

where $\phi_j^F(x_i) = b_j^{F-1}\left(b_i^F(x_i)\right)$.

Dividing yields

$$\begin{aligned}&\frac{b_i^{F\prime}(x_i)}{b_j^{F\prime}\left(\phi_j^F(x_i)\right)}\\&=\frac{s_{ii}f(x_i)F\left(\phi_j^F(x_i)\right)}{s_{jj}f\left(\phi_j^F(x_i)\right)F(x_i)}\\&\times\frac{s_{jj}v\left(\phi_j^F(x_i),x_i\right)-s_{ji}v\left(x_i,\phi_j^F(x_i)\right)-(s_{jj}-s_{ji})b_j^F\left(\phi_j(x_i)\right)}{s_{ii}v\left(x_i,\phi_j^F(x_i)\right)-s_{ij}v\left(\phi_j^F(x_i),x_i\right)-(s_{ii}-s_{ij})b_i^F(x_i)}.\end{aligned}$$

Note that in equilibrium, $b_i^F(\overline{x})=b_j^F(\overline{x})<v(\overline{x},\overline{x})$. Then, for $x_i=\overline{x}$, $\phi_j(x_i)=\overline{x}$, and we have

$$\frac{b_i^{F\prime}(\overline{x})}{b_j^{F\prime}(\overline{x})}=\frac{s_{ii}}{s_{jj}}\frac{(s_{jj}-s_{ji})}{(s_{ii}-s_{ij})}=\frac{1-s_j}{1-s_i}<1$$

and hence

$$b_i^F(x)>b_j^F(x)\text{ for }x\text{ close to }\overline{x}.$$

Now, suppose $b_i^F(\widehat{x})=b_j^F(\widehat{x})$ for some $\widehat{x}\in(\underline{x},\overline{x})$, (and hence $\widehat{x}=\phi_j^F(\widehat{x})$) then

$$\frac{b_i^{F\prime}(\widehat{x})}{b_j^{F\prime}(\widehat{x})}=\frac{1-s_j}{1-s_i}<1$$

i.e.

$$b_i^{F\prime}(\widehat{x})<b_j^{F\prime}(\widehat{x})$$

and hence

$$\frac{b_i^F(\widehat{x})}{b_j^F(\widehat{x})}\text{ is strictly decreasing at }\widehat{x}$$

since

$$\frac{d}{dx}\frac{b_i^F(x)}{b_j^F(x)}=\frac{b_i^{F\prime}(x)-b_j^{F\prime}(x)}{b_j^F(x)}<0\text{ at }\widehat{x}.$$

It follows that $b_i^F(x)<b_j^F(x)$ for all $x\in(\widehat{x},\overline{x})$, which contradicts $b_i(x)>b_j(x)$ for x close to $\overline{x}$. Thus, $b_i^F(x)>b_j^F(x)$ for all $x\in(\underline{x},\overline{x})$.

Proof of Proposition 8

Consider the second-price auction with private value, and suppose $s_{ij} = 0$. Then, for bidder i

$$\Pi_i(b; x_i) = s_{ii} \int_{\underline{x}}^{b_j^{-1}(b)} (x_i - b_j(\alpha)) f(\alpha) d\alpha$$

and hence

$$b_i(x_j) = x_i.$$

For bidder j,

$$\Pi_j(b; x_j) = s_{jj} \int_{\underline{x}}^{b} (x_j - \alpha) f(\alpha) d\alpha + s_{ji} \int_{b}^{\overline{x}} (\alpha - b) f(\alpha) d\alpha.$$

The first-order condition is:

$$s_{jj}(x_j - b) f(b) - s_{ji} \int_{b}^{\overline{x}} f(\alpha) d\alpha = 0$$

or

$$(x_j - b) f(b) - s_j [1 - F(b)] = 0.$$

Let $F(x) = x$, then

$$(x_j - b) - s_j(1 - b) = 0$$

or

$$b_j(x_j) = \frac{x_j - s_j}{1 - s_j}$$

for $x_j \geq s_j$ (and for $x_j < s_j$, there is no bid). Observe that $b_j(x_j)$ is decreasing in s_j and $b_j(x) < b_i(x)$. That is, the bidder with more cross-holding will bid less aggressively.

Proof of Proposition 9

We have

$$b_j^{S\prime}\left(\phi_j^S(x_i)\right) = \left[\frac{s_{ii}\left[x_i - b_i^S(x_i)\right] - s_{ij}\left[\phi_j^S(x_i) - b_i^S(x_i)\right]}{s_{ij}}\right] \times \frac{f\left(\phi_j^S(x_i)\right)}{1 - F\left(\phi_j^S(x_i)\right)} \tag{25}$$

and

$$b_i^{S\prime}(x_i) = \left[\frac{s_{jj}\left[\phi_j^S(x_i) - b_j^S\left(\phi_j^S(x_i)\right)\right] - s_{ji}\left[x_i - b_j^S\left(\phi_j^S(x_i)\right)\right]}{s_{ji}} \right] \times \frac{f(x_i)}{1-F(x_i)} \tag{26}$$

where $\phi_j^S(x_i) = b_j^{S-1}\left(b_i^S(x_i)\right)$. Dividing, we get

$$\frac{b_i'(x_i)}{b_j'(\phi_j(x_i))} = \frac{[s_{jj}\phi_j(x_i) - s_{ji}x_i] - (s_{jj} - s_{ji})b_i(x_i)}{[s_{ii}x_i - s_{ij}\phi_j(x_i)] - (s_{ii} - s_{ij})b_j(\phi_j(x_i))\}} \frac{s_{ij}}{s_{ji}} \frac{f(x_i)}{f(\phi_j(x_i))} \frac{1-F(\phi_j(x_i))}{1-F(x_i)}$$

Using the L'Hopital rule, we have

$$\begin{aligned}\phi_j'(\underline{x}) &= \frac{s_{jj}\phi_j'(\underline{x}) - s_{ji} - (s_{jj} - s_{ji})b_i'(\underline{x})}{s_{ii} - s_{ij}\phi_j'(\underline{x}) - (s_{ii} - s_{ij})b_j'(\underline{x})\phi_j'(\underline{x})} \frac{s_{ij}}{s_{ji}} \\ &= \frac{s_{jj}\phi_j'(\underline{x}) - s_{ji}}{s_{ii} - s_{ij}\phi_j'(\underline{x})} \frac{s_{ij}}{s_{ji}} \\ &= \frac{\phi_j'(\underline{x}) - s_j}{1 - s_i\phi_j'(\underline{x})} \frac{s_i}{s_j}\end{aligned}$$

and hence

$$\phi_j'(\underline{x}) = \frac{s_j - s_i + \sqrt{\left(s_j - s_i\right)^2 + 4s_i^2 s_j^2\right)}}{2s_i s_j} > 1$$

when $s_j > s_i$. That is, $b_i'(\underline{x}) > b_j'(\underline{x})$ and hence $b_i(x) > b_j(x)$ for x sufficiently close to $\underline{x}$.

Proof of Proposition 10

From the first-order conditions (15) and (16),

$$b_j'(\phi_j(x_i)) = (1 - s_i)[v(\phi_j(x_i), x_i) - b_i(x_i)] \frac{1}{\phi_j(x_i)}$$

and

$$b_i'(x_i) = (1 - s_j)[v(\phi_j(x_i), x_i)) - b_j(\phi_j(x_i))] \frac{1}{x_i}.$$

Dividing, we get

$$\frac{b_i'(x_i)}{b_j'(\phi_j(x_i))} = \frac{1 - s_j}{1 - s_i} \frac{\phi_j(x_i)}{x_i}$$

Since $b_i'(x_i) = b_j'(\phi_j(x_i))\phi_j'(x_i)$, we have

$$\phi_j'(x_i) - \frac{1-s_j}{1-s_i}\frac{1}{x_i}\phi_j(x_i) = 0$$

and hence

$$\phi_j(x_i) = x_i^{\frac{1-s_j}{1-s_i}}.$$

Therefore, integrating the first-order condition (16), we get:

$$b_i(x_i) = (1-s_j)\frac{\int_0^{x_i} v\left(t, t^{\frac{1-s_j}{1-s_i}}\right) t^{-s_j}\,dt}{{x_i}^{1-s_j}}.$$

Similarly, we can derive

$$b_j(x_j) = (1-s_i)\frac{\int_0^{x_j} v\left(t, t^{\frac{1-s_i}{1-s_j}}\right) t^{-s_i}\,dt}{{x_j}^{1-s_i}}.$$

Note that for $s_i < s_j$, $x < x^{\frac{1-s_j}{1-s_i}} = \phi_j(x)$ and hence

$$b_i(x) = b_j\left(x^{\frac{1-s_j}{1-s_i}}\right) > b_j(x).$$

Proof of Proposition 11

Recall

$$b_i'(x_i) = \frac{1-s_j}{s_j}\frac{1}{1-x_i}[v(\phi_j(x_i), x_i) - b_j(\phi_j(x_i))]$$

and

$$b_j'(\phi_j(x_i)) = \frac{1-s_i}{s_i}\frac{1}{1-\phi_j(x_i)}[v(\phi_j(x_i), x_i) - b_i(x_i)].$$

Using an argument similar to the one given for Proposition 10, we obtain

$$\phi_j'(x_i) = \frac{1-s_j}{1-s_i}\frac{s_i}{s_j}\frac{1-\phi_j(x_i)}{1-x_i}.$$

Then, integration now yields,

$$\phi_j(x) = 1 - (1-x)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}}$$

and

$$b_i(x_i) = \frac{1-s_j}{s_j}\frac{\int_0^{x_i} v(t, \phi_j(t))(1-t)^{-\frac{1}{s_j}}\,dt}{(1-x_i)^{-\frac{1-s_j}{s_j}}}.$$

When $s_i < s_j$we have

$$\frac{1-s_j}{1-s_i}\frac{s_i}{s_j} < 1$$

and hence

$$x > 1-(1-x)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}} = \phi_j(x)$$

and

$$b_i(x) = b_j\left(1-(1-x)^{\frac{1-s_j}{1-s_i}\frac{s_i}{s_j}}\right) < b_j(x).$$

Derivation of expected revenue with asymmetric cross-holding

First, note that when $v(t,t) = t + t$, direct substitution yields

$$b_i^F(x_i) = \frac{1-s_i}{2-s_i}x_i^{\frac{1-s_j}{1-s_i}} + \frac{1-s_j}{2-s_j}x_i.$$

Then, the expected reveune is

$$\begin{aligned} &R^F \\ &= \int_0^1\int_0^{\phi_j(x_i)} b_i(x_i)dx_jdx_i + \int_0^1\int_{\phi_j(x_i)}^1 b_j(x_j)dx_jdx_i \\ &= \frac{(2-s_i-s_j)\left(12+5s_i^2+5s_j^2+14s_is_j-15s_i-15s_j-3s_i^2s_j-3s_is_j^2\right)}{(2-s_i)(2-s_j)(3-s_i-2s_j)(3-2s_i-s_j)}. \end{aligned}$$

Similarly, in the second-price auction, one can derive

$$\begin{aligned} R^S &= \int_0^1\int_0^{\phi_j(x_i)} b_j(x_j)dx_jdx_i + \int_0^1\int_{\phi_j(x_i)}^1 b_i(x_i)dx_jdx_i \\ &= \frac{(1-s_i)(1-s_j)\left(s_i^2+s_j^2+4s_is_j+6s_i^2s_j^2-6s_i^2s_j-6s_is_j^2\right)}{(1-s_is_j)(s_i+2s_j-3s_is_j)(2s_i+s_j-3s_is_j)}. \end{aligned}$$

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